Refinements of the Solution Theory for Singular SPDEs

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Abstract

This thesis is concerned with the study of singular *stochastic partial differential equations* (*SPDE*s). We develop extensions to existing solution theories, present fundamental interconnections between different approaches and give applications in financial mathematics and mathematical physics.

SPDEs are partial differential equations which are modified by some stochastic term, representing "noise". Due to this random term the solution of the equation becomes stochastic and typically exhibits only little regularity ("smoothness"). The equation is called singular if the expected regularity of the solution is so poor that the equation cannot be solved with tools from classical analysis. The past few years have seen remarkable breakthroughs in the study of such systems, especially due to the application of *paracontrolled* methods [GIP15] and the development of the theory of *regularity structures* [Hai14].

We present a new method for discrete approximations of singular SPDEs. At the core of the theories above lies the idea, that a description of the solution and the noise at small scales yields a better understanding of their interplay, which allows in many cases for a solution of the equation. This motivates the study of discrete approximations of singular SPDEs on lattices with small mesh size. If the latter tends to zero one expects to obtain the continuous analogue in the limit. We develop a discrete modification of the paracontrolled methods that allow for an investigation of such problems. As an example we study branching random walks on (Bravais) lattices in a random environment. We allow for a reproduction rate that grows non-linearly in the number of particles. When jumping to the continuum we find a universal pattern: The nonlinearity vanishes in the limit and the equation reduces to the linear parabolic Anderson model.

We further show in this thesis that there is a fundamental interconnection between the paracontrolled Ansatz and the theory of regularity structures. The latter is based on a local Taylor-like expansion, while in the paracontrolled framework a frequency modulation (the *paraproduct*) is subtracted to smoothen the solution. We here prove that there is in fact a fundamental symmetry between both concepts: It is possible to locally expand a function (or distribution) if and only if the components of this expansion can be smoothened by paraproducts. This corresponds to a description of the spaces of modelled distributions from [Hai14] via Fourier methods, in quite a similar fashion as one can describe Hölder spaces using a Littlewood-Paley decomposition. We apply this correspondence to give a conceptual new proof of Schauder estimates in the framework of regularity structures, based on paraproducts.

We further consider in this thesis two applications of the solution theory for singular SPDEs. We demonstrate the power of the theory of regularity structures by presenting an application to mathematical finance: We develop a theory for

robust approximations of option prices under rough volatility.

Moreover, we show that the stochastic Schrödinger equation with non-periodic noise possesses solutions. Our proof is based on the key observation that the solutions to this equation remain localized on finite time intervals.

Zusammenfassung

Diese Dissertation widmet sich der Untersuchung singulärer *stochastischer partieller Differentialgleichungen* (engl. *SPDE*s). Wir entwickeln Erweiterungen der bisherigen Lösungstheorien, zeigen fundamentale Beziehungen zwischen verschiedenen Ansätzen und präsentieren Anwendungen in der Finanzmathematik und der mathematischen Physik.

SPDEs sind partielle Differentialgleichungen, die durch einen stochastischen Rauschterm ergänzt werden. Durch den Zufallsterm in der Gleichung ergibt sich eine stochastische Lösung, die typischerweise nur wenig Regularität („Glattheit“) besitzt. Man nennt die Gleichung singulär, falls die zu erwartende Regulärität der Lösung so niedrig ist, dass die Gleichung mit Methoden der klassischen Analysis nicht sinnvoll gelöst werden kann. In den letzten Jahren kam es zu bahnbrechenden Erkenntnissen in der Untersuchung solcher Systeme, vor allem durch die Anwendung *parakontrollierter* Methoden [GIP15] und durch die Entwicklung der Theorie der *Regularitätsstrukturen* [Hai14].

Wir präsentieren eine Methode zur diskreten Approximierung singulärer SPDEs. Kern-idee der oben genannten Theorien ist die Überlegung, dass eine Beschreibung der Lösung und des Rauschens auf kleinen Skalen dazu benutzt werden kann deren Wechselwirkung zu verstehen, was letztendlich in vielen Fällen die Lösung der Gleichung ermöglicht. Dies motiviert die Approximation solcher Relationen durch diskrete Systeme auf Gittern mit kleiner Gitterkonstante. Wenn letztere gegen Null strebt, sollte sich die kontinuierliche Gleichung im Limes ergeben. Wir entwickeln eine diskrete Abwandlung der parakontrollierten Methoden, die es erlaubt derartige Fragestellungen zu untersuchen. Als Beispiel studieren wir sich verzweigende Irrfahrten auf einem (Bravais-)Gitter in einer Zufallsumgebung. Wir nehmen an dass die Reproduktionsrate nichtlinear in der Teilchenzahl wächst. Beim Sprung ins Kontinuum finden wir eine universelle Gesetzmäßigkeit: Die Nichtlinearität verschwindet und als Grenzwert ergibt sich stets das lineare parabolische Anderson-Modell.

Wir zeigen weiterhin eine fundamentale Beziehung zwischen dem parakontrollierten Ansatz und der Theorie der Regularitätsstrukturen auf. Während in letzterer die Lösung mittels einer lokalen („Taylor“-)Entwicklung untersucht wird, beruht der parakontrollierte Ansatz auf einer Glättung duch Subtraktion einer Frequenzmodulation (des *Paraprodukts*). Wir zeigen hier, dass eine fundamentale Symmetrie zwischen beiden Konzepten besteht: Eine Funktion (bzw. Distribution) lässt sich lokal genau dann entwickeln, falls die Komponenten dieser Entwicklung sich durch Paraprodukte glätten lassen. Dies entspricht einer Charakterisierung der Räume der modellierten Distributionen aus [Hai14] mittels Fourier-Methoden, ähnlich der Beschreibung von Hölderräumen durch Littlewood-Paley-Zerlegung. Als Anwendung dieser Korrespondenz geben wir einen konzeptuell neuen Beweis der Schauder-Abschätzungen für Regularitätsstrukturen, basierend auf Paraprodukten.

Wir demonstrieren zudem die Mächtigkeit der Theorie der Regularitätsstrukturen mittels der Entwicklung einer Theorie für die robuste Approximation von Optionspreisen unter dem Einfluss rauer Volatilität. Des Weiteren zeigen wir, dass die Schrödinger-Gleichung mit nicht-periodischem Rauschterm unter geeigneten Vorraussetzungen eine Lösung besitzt und zeigen, dass diese auf endlichen Zeitskalen lokalisiert bleibt.

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To Laura
and my parents

Contents

Chapter 1

Introduction

As mathematicians we like to think that our discipline provides the language "in which the book of nature is written", to cite a famous tuscan mathematician [Gal23]. Many generations of scientists have found an immense variety of fundamental laws that govern the course of events in our cosmos, many of them condensed in some partial differential equation. However, as probably every experimental physicist can confirm, we can often only apply this knowledge in a very protected environment. Much of the effort put into a physical experiment serves often only one purpose: lock out the surrounding world. In "real life" every physical process is affected by such a vast number of impacts that the relations we can summarize in a formula only hold, if at all, in average. In other words: We live in a noisy world!

The concept of a stochastic partial differential equation (SPDE) can be seen as an attempt to develop models that take this fact into account. In a nutshell, the idea is to disturb some (deterministic) partial differential equation by a stochastic object, called the "noise". The unspoken assumption on the model is that the sheer number of impacts force this object, by a central limit type argument, to be governed by a certain probabilistic law, which is in essence all one really has to know about the noisy background of the studied system.

Let's start with the possibly easiest SPDE of all. The archetype of a partial differential equation is the heat equation on $[0,T] \times \mathbb{R}^d$

$$(\partial_t - \Delta)u = 0,\ u|_{t=0} = u_0\,. \tag{1.1}$$

It describes the time evolution of a heat profile u on $\mathbb{R}^d$, starting from u_0 at time $t = 0$. We can turn this equation into a SPDE by adding a random "forcing" term ξ to the right hand side

$$(\partial_t - \Delta)u = \xi,\ u|_{t=0} = u_0\,. \tag{1.2}$$

In dimension $d = 2$ one could think of (1.1) as modelling the heat flow in some flat layer in an isolated enviroment, while in (1.2) this layer might be placed in some ambient gas with fluctuating temperature. Through ξ randomness enters the equation, in particular the solution u of (1.2) can now be read as a random variable/stochastic process.

A natural choice for ξ might be a random force that models fluctuations in time and space. Namely, take a cylindrical Brownian motion $W = (W(t))_{t\in[0,T]}$ on $L^2(\mathbb{R}^d)$ (see for example [DPZ02] or for a gentle introduction [Hai09]), and set

$$\xi = \frac{\mathrm{d}}{\mathrm{d}t} W \,. \tag{1.3}$$

Of course W is almost surely nowhere differentiable on $[0,T]$, so that we have to say what we mean by (1.3). One possible way is to multiply formally both sides of (1.1) by $\mathrm{d}t$, so that we obtain the (infinite dimensional) stochastic differential equation

$$\mathrm{d}u = \Delta u \,\mathrm{d}t + \mathrm{d}W,\; u|_{t=0} = u_0 \,. \tag{1.4}$$

The fundamental solution to the homogeneous problem (1.1) is given by $G(t,x) = \mathbf{1}_{t>0}\frac{1}{(4\pi t)^{d/2}}\, e^{-\frac{|x|^2}{4t}}$, so that by Duhamel's principle we can give the solution to (1.4) as

$$u(x) = (e^{t\Delta}u_0)(x) + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y)\,\mathrm{d}W(s,y)\,, \tag{1.5}$$

where we wrote $e^{t\Delta}u_0 = G(t,\cdot) *_{\mathbb{R}^d} u_0$. There is also an "analytical way" to write this solution. We can see the identity (1.3) as a distributional derivative or, equivalently, we set for Schwartz functions $\varphi \in \mathcal{S}(\mathbb{R}^{d+1})$

$$\xi(\varphi) := \int_{[0,T]\times\mathbb{R}^d} \varphi \,\mathrm{d}W \,.$$

With this notation one may write (1.5) as

$$u(t,x) = (e^{t\Delta}u_0)(x) + (G *_{\mathbb{R}^{d+1}} \xi)(t,x) \tag{1.6}$$

(of course one has to be careful that this convolution makes sense, compare Section 6.1 below). Note that the random distribution ξ satisfies by construction and the Itô isometry for cylindrical Brownian motion $\xi(\varphi) \overset{\mathrm{d}}{\sim} \mathcal{N}(0, \|\varphi\|^2_{L^2([0,T]\times\mathbb{R}^d)})$, where $\mathcal{N}(0, \|\varphi\|^2_{L^2([0,T]\times\mathbb{R}^d)})$ denotes the normal distribution with mean 0 and variance $\|\varphi\|^2_{L^2([0,T]\times\mathbb{R}^d)}$. We call ξ the white noise on $[0,T]\times\mathbb{R}^d$. This allows to treat ξ which do not depend on time, but only on space. Namely, choose ξ as white noise on $\mathbb{R}^d$, that is a random distribution such that for $\varphi \in \mathcal{S}(\mathbb{R}^d)$

$$\xi(\varphi) \overset{\mathrm{d}}{\sim} \mathcal{N}(0, \|\varphi\|_{L^2(\mathbb{R}^d)})\,. \tag{1.7}$$

While we can still solve (1.2) via (1.6), there is no "stochastic" version in the sense of (1.5), as there is no cylindrical Brownian motion W in the background. This shortage gets us into trouble when we consider a multiplicative noise term instead:

$$(\partial_t - \Delta)u = u \cdot \xi,\ u|_{t=0} = u_0\,. \tag{1.8}$$

This equation is called the *multiplicative stochastic heat equation* for space-time white noise ξ and the *parabolic Anderson model (PAM)* for white noise in space. While for space-time white noise $\xi = \frac{\mathrm{d}}{\mathrm{d}t}W$ and $d = 1$ one can still solve (1.8) via its stochastic formulation

$$u(t,x) = (e^{t\Delta}u_0)(x) + \int_0^t \int_{\mathbb{R}} G(t-s, x-y)\, u(y)\, \mathrm{d}W(s,y)\,,$$

compare [DPZ02, Theorem 7.3.5], one has for white noise $\xi \in \mathcal{S}'(\mathbb{R}^d)$ in space, i.e. the parabolic Anderson model, no choice but to work with the "analytical" formulation of (1.8), which reads as

$$u(t,x) = (e^{t\Delta}u_0)(x) + \int_0^t \left(e^{(t-s)\Delta}(u\cdot\xi)\right)(x)\,\mathrm{d}s\,. \tag{1.9}$$

Consider this equation in dimension $d = 2$. By construction ξ is a (random) distribution, while there is not much hope that u is smooth. In fact, by classical Schauder theory one sees that the best one can hope for is that u is Hölder continuous with exponent $\gamma < 1$. As Schwartz already remarked in his magnum opus [Sch50, Section 5.1] a product between a distribution and a non-smooth function is in general not defined. Although there is a way to define products of this type if the regularity of both factors is not too poor [BCD11, Theorem 2.85] (or Corollary 2.1.35 below), it turns out that this just fails to be the case for the expected Hölder regularity of u. In other words, we do not even know in which sense there might be a solution to (1.9). A stochastic partial differential equation that exhibits such ill-defined terms is known as a *singular SPDE*.

Obstacles of this kind were overcome for the first time only fairly recently by two different techniques. In [GIP15] the authors use tools of paracontrolled calculus, developed by Bony [Bon81], to give a rigorous meaning to (1.8) in $d = 2$ (on a torus). In [BBF17] a modification of this theory was applied to solve the equation for $d = 3$. Another approach, the theory of regularity structures, was presented in [Hai14]. With the help of this theory (1.8) could be solved with white noise in two and three dimensional space [Hai14, HL16] and with space-time white noise with $d = 1$ in the analytical formulation (1.9), compare [HP15, HL16]. Let us bookkeep some typical examples of singular SPDEs which were solved with the theories layed out in [Hai14] and [GIP15]. By solved we mean here that a solution was given in the "analytical formulation" such as 1.9.

- The generalized parabolic Anderson model (*gPAM*)

$$(\partial_t - \Delta)u = F(u)\xi \tag{gPAM}$$

on $[0, T] \times \mathbb{R}^d$ with white noise in space $\xi \in \mathcal{S}'(\mathbb{R}^d)$ in $d = 2, 3$, was solved in a periodic setup by [GIP15, Hai14, HP15, BBF17] and on the full space by [HL16, MP17].

- The Kardar-Parisi-Zhang equation (KPZ)

$$(\partial_t - \Delta)h = (\partial_x h)^2 + \xi \tag{KPZ}$$

on $[0, T] \times \mathbb{R}$, where ξ is space-time white noise, was solved (in a periodic setup) by [Hai13] (using rough path theory) and [Hai14, GP15b, KM17].

- The three-dimensional stochastic quantization equation (Φ^4_3)

$$(\partial_t - \Delta)\phi = -\phi^3 + \xi \tag{Φ^4_3}$$

on $[0, T] \times \mathbb{R}^3$, where ξ is space-time white noise, solved by [CC13, Hai14, MW17a, ZZ15, Kup16].

- The stochastic Burgers equation

$$(\partial_t - \Delta)u = F(u)\partial_x u + \xi \tag{SBE}$$

on $[0, T] \times \mathbb{R}$ with space-time white noise, solved in [Hai14, GIP15]. A more singular version, with the derivative of white noise as a noise term, was solved in [GP15b].

- The stochastic nonlinear Schrödinger equation

$$(\imath\partial_t - \Delta)u = \lambda|u|^{2\sigma}u + u\xi \tag{SNLS}$$

on $[0, T] \times \mathbb{R}^2$ with white noise in space, was solved by [DW16, DM17].

In some references an even more elaborate version of the corresponding SPDE is considered. All of these equations have one feature in common: They contain one term which is ill-defined since it contains a product of a distribution with a non-smooth function. We marked the corresponding term in blue. Although, as already pointed out above, such a product can be defined with tools of classical analysis if the smoothness of the considered objects is not too bad, all these equations fall into a regime where this is not possible without further work.

The development that lead to the theories in [Hai14] and [GIP15] was initialized by ideas from rough path theory. It was realized by T. Lyons [Lyo91, Lyo98] that for a robust definition of stochastic integrals one needs to enhance the driving noise with additional information. Gubinelli [Gub04] then proposed to consider integrands that behave on small scales like this augmented integrator. Most of this theory is purely deterministic, only the enhancement of the driving noise is usually achieved by applying probabilistic tools. Both, the theory of regularity structures and the paracontrolled approach, adapt these ideas to a problem such as (1.9). In order to define the product $u \cdot \xi$ one first equips ξ with an improved structure, by using its stochastic nature, and then gives a local description of u in terms of this enhanced noise, for which one uses that u solves the considered equation. The enhancement of ξ has a similar taste in both concepts, although for the theory in [Hai14] one can apply a universal algebraic machinery for this task [BHZ16, CH16]. In this step one often has to subtract divergent terms in the considered equation in order to give a meaning to the ill-defined products. This step is known as *renormalization*.

The local description of the solution follows different philosophies in [Hai14] and [GIP15]. The regularity structure framework builds on local Taylor-like expansions and can be seen as a far reaching generalization of the theory of rough paths [FV10, FH14] and of Stein's notion of differentiability [Ste70]. An object that can be expanded in this way is called a *modelled distribution*. The paracontrolled approach shows that the subtraction of a frequency modulation (the paraproduct) smooths the solution, which can be understood as the cancellation of fluctuations on small scales. A distribution which can be regularized by such a procedure is a *paracontrolled distribution*. In spite of their conceptual disparity it has been conjectured [GIP15, p. 54] that a one-to-one correspondence between paracontrolled distributions and modelled distributions might exist.

We here show that this conjecture is true. We further give a toolbox for a discrete approximation of singular SPDEs such as the ones mentioned above in the paracontrolled setup. We also present an application of regularity structures to option pricing and show how (SNLS) can be solved on the full space $\mathbb{R}^2$.

In Chapter 2 we give the fundamental definitions on which this thesis is build. This includes a Fourier theory which allows for weighted (and anisotropic) Besov spaces and the definitions and fundamental results from regularity structures. The definition and a few basic properties of white noise are recalled as well.

In Chapter 3 we translate the paracontrolled calculus to a discrete framework. Many results from the continuous theory translate into a setup on Bravais lattices. The presented methods are forged to study the convergence of singular SPDEs on a refinining sequence of lattices. Let us highlight also Section 3.4, where we provide a quite universal apparatus for Wick renormalization on such a sequence. Chapter 4 demonstrates the power of the tools developed in Chapter 3: We prove a weak universality result for the parabolic Anderson model. For a discrete version of gPAM on a Bravais lattice with mesh size ε we show in the limit $\varepsilon \to 0$ that the equation and its solutions scale to the *linear* parabolic Anderson model (1.8), so that non-linearity on small scales becomes invisible in the big picture. Both chapters, 3 and 4, are based on [MP17].

Chapter 5 shows that there is a complete correspondence between the modelled distributions from regularity structures and paracontrolled distributions. For this purpose we show how one can define a paraproduct on a regularity structure equipped with a model. We also present a suitable definition of the used spaces that allows for a possible blowup at time $t = 0$ and prove some basic properties. We further give some Whitney type extension result for modelled distributions. Chapter 6 exemplifies how Schauder estimates for modelled distributions can be proved with the correspondence proved in chapter 5. The derivation of Schauder estimates is one of the most challenging parts in [Hai14]. We show a different method that uses paraproducts instead. This will be content of [MP18].

In Chapter 7 we show how the theory of regularity structure can be applied to a problem arising from financial mathematics. In [BFG16] a formula for option pricing under rough volatility was proposed. We develop a regularity structure for the stochastic integrals of fractional Brownian motion in this formula and present a robust approximation mechanism. This is based on [BFG$^+$17]

The solution theory for (SNLS) is, although similar in its philosophy, in some aspects perpendicular on those for the other singular SPDEs presented in this introduction. The involved semigroup $e^{-\imath t\Delta}$ does not show any smoothing properties, which is in high contrast to the properties of the heat semigroup. However, it was shown in [DW16] that an application of a transformation from [HL15] can still yield a solution for this equation on a torus $\mathbb{T}^2$. We show in Chaper 8 that one can in fact construct a solution on $\mathbb{R}^2$ by studying the localization properties of the stochastic Schrödinger equation. This is based on [DM17].

Chapter 3 & 4: Discrete paracontrolled calculus on Bravais lattices & weak universality of the parabolic Anderson model

We consider the following discrete version of (gPAM) in $d = 2$:

$$(\partial_t - L_\mu^{\mathcal{G}})\phi = F(\phi)\eta, \qquad \phi(0) = \frac{1}{|\mathcal{G}|}\mathbf{1}_{\cdot=0} \tag{1.10}$$

on a Bravais lattice $\mathcal{G} = \mathbb{Z}a_1 + \mathbb{Z}a_2$, i.e. the set of integer combinations of linear independent vectors a_1, a_2. We write $|\mathcal{G}|$ for the size of a “unit cell”, so that $|\mathcal{G}| = \det(a_1, a_2)$. Here $L_\mu^{\mathcal{G}}$ is the generator of a symmetric, time-continuous random walk with jump rates $\mu(y - x)$ from $x \in \mathcal{G}$ to $y \in \mathcal{G}$, $(\eta(z))_{z\in\mathcal{G}}$ is a family of independent random variables with enough moments and $F \in C^2(\mathbb{R};\mathbb{R})$ has a bounded second derivative and satisfies $F(0) = 0$. Equation (1.10) can be seen as describing the average of a branching random walk on $\mathcal{G}$ in some random environment η. We are interested in the behavior of (1.10) on large scales ε^{-1}, more precisely we consider $u^\varepsilon(t,x) = \varepsilon^{-2}\phi(\varepsilon^{-2}t, \varepsilon^{-1}x)$. It turns out one can get a meaningful expression in the limit $\varepsilon \to 0$ if one requires

$$\mathbb{E}[\eta(z)] = -F'(0)c^\varepsilon\varepsilon^2,\ \mathrm{Var}(\eta(z)) = \frac{\varepsilon^2}{|\mathcal{G}|}$$

so that η and ϕ in (1.10) actually depend on ε, which we suppressed in the notation. Here $c^\varepsilon \approx \log(\varepsilon^{-1})$ is some specific deterministic constant. (1.10) can then be reshaped to

$$(\partial_t - L^{\mathcal{G}^\varepsilon}_{\mu^\varepsilon})u^\varepsilon = F^\varepsilon(u^\varepsilon)(\xi^\varepsilon - F'(0)c^\varepsilon_\mu),\ u^\varepsilon(0) = \frac{1}{|\mathcal{G}^\varepsilon|}\mathbf{1}_{\cdot=0}\,, \tag{1.11}$$

on $\mathcal{G}^\varepsilon = \varepsilon\mathcal{G} = \mathbb{Z}\varepsilon a_1 + \mathbb{Z}\varepsilon a_2$, where $\xi^\varepsilon(z) := \varepsilon^{-2}(\eta(\varepsilon^{-1}z) + F'(0)c^\varepsilon_\mu\varepsilon^2)$ for $z \in \mathcal{G}^\varepsilon$, $F^\varepsilon = \varepsilon^{-2}F(\varepsilon^2\cdot)$ and where $L^{\mathcal{G}^\varepsilon}_{\mu^\varepsilon}$ is the generator of a random walk on $\mathcal{G}^\varepsilon$ with rates $\mu^\varepsilon(y-x) = \varepsilon^{-2}\mu(\varepsilon^{-1}(y-x))$ for $x,\, y \in \mathcal{G}^\varepsilon$. Using the definition of F^ε one might use a Taylor expansion around 0 to estimate

$$F^\varepsilon(u^\varepsilon) = 0 + F'(0)u^\varepsilon + \ldots$$

so that the limit of (1.11) can be guessed as

$$(\partial_t - L^{\mathbb{R}^2}_\mu)u = F'(0)u(\xi - F'(0)\infty),\ u(0) = \delta_0\,, \tag{1.12}$$

where $L^{\mathbb{R}^2}_\mu$ is the limiting generator (compare Definition 3.3.3), which is simply an elliptic partial differential operator of second order. The symbol "$-\infty$" indicates that in order to define the product $u\cdot\xi$ in (1.12) one has to introduce a renormalization as shortly mentioned above, in our case one could see this as indicating the divergent sequence $c^\varepsilon_\mu \approx \log(\varepsilon^{-1})$ we had to introduce (although we didn't say why yet).

From the nonlinearity F that we put into our system only the value $F'(0)$ remains visible in the limit $\varepsilon \to 0$. The linear parabolic Anderson model (1.12), which is essentially the same as (1.8), has therefore a privileged role in the large scale description of branching random walks. We call this observation *the weak universality for the parabolic Anderson model* in dimension 2.

The rigorous proof of the steps above is the content of Chapter 4 and we formulate our main result in Theorem 4.3.6. Our observations join the ranks of similar findings for other singular SPDEs, compare for example [HQ15, GJ14, GP15a, GP16] and [MW17b, HX16, SW16, GKO17, OT17].

The tools we apply to show the weak universality of the parabolic Anderson model in dimension 2 are derived from ideas from paracontrolled calculus [GIP15, GP15b]. In previous considerations of the limit of discrete singular SPDEs in this framework [CGP17, GP15b, ZZ15] the solution u^ε was first extended to the full space and then paracontrolled methods were applied to the (continuous) equation of the extended solution $\mathscr{E}^\varepsilon u^\varepsilon$. However this idea has the drawback that it leads to additional operators in the equation, which make the analytical handling quite intricate. We choose therefore to go a different route. We show how paracontrolled techniques can be directly applied to the discrete equations to derive a priori bounds in discrete spaces. The extension $\mathscr{E}^\varepsilon u^\varepsilon$ will only be considered in the very last step, that is in the passage to the limit $\varepsilon \to 0$. As in the limit the operator $\mathscr{E}^\varepsilon$ commutes with all operations in the equation, it follows at once that the limit of $\mathscr{E}^\varepsilon u^\varepsilon$ satisfies the desired relation. We present these methods in Chapter 3. Since the corresponding techniques can in principle be used for any equation for which paracontrolled techniques apply, Chapter 3 is a self-contained presentation in its own right. A discrete version of the Schauder estimates for parabolic singular SPDEs as in [GIP15, GP15b] is given in Chapter 4. For similar approaches in the context of regularity structures compare [HM15, CM16].

In Chapter 3 the Fourier transform of discrete functions $f : \mathcal{G} \to \mathbb{C}$ is studied. As in the case of $\mathcal{G} = \mathbb{Z}^2$ the Fourier transform $\mathcal{F}_\mathcal{G} f$ turns out to be a periodic function (or distribution). As domain of $\mathcal{F}_\mathcal{G} f$ we take a certain parallelotope $\widehat{\mathcal{G}}$, centered at 0, called in this work the "Fourier cell". In the case of $\mathcal{G} = \mathbb{Z}^2$ this simply coincides with the well-known torus $\widehat{\mathcal{G}} = \mathbb{T}^2 = [-1/2, 1/2)^2$. We are actually interested in a sequence $\mathcal{G}^\varepsilon = \varepsilon \cdot \mathcal{G}$. The Fourier cell then scales like

$$\widehat{\mathcal{G}^\varepsilon} = \varepsilon^{-1}\widehat{\mathcal{G}},$$

so that morally for $\varepsilon = 0$ the Fourier transform is defined on the full space and periodicity becomes invisible. In Section 3.1 we perform a Littewood-Paley decomposition of $\widehat{\mathcal{G}^\varepsilon}$. By boundedness of the Fourier cell only finitely many blocks are required. More precisely, for $\varepsilon \approx 2^{-N}$ the number of blocks is of order N. One can then define paraproducts, resonance products and commutators just as in [GIP15]. As a discrete function space we define a "discrete Besov space" $\mathcal{B}^\gamma_{p,q}(\mathcal{G}^\varepsilon)$. The topology of $\mathcal{B}^\gamma_{p,q}(\mathcal{G}^\varepsilon)$ for a fixed ε is rather uninteresting, it coincides with the one of $\ell^p(\mathcal{G}^\varepsilon)$, but what we are really after are estimates in these spaces which are uniform in ε. For such bounds we can apply a discrete extension operator $\mathscr{E}^\varepsilon$ from $\mathcal{G}^\varepsilon$ to $\mathbb{R}^2$, compare Lemma 3.1.10, to get a bounded sequence in $\mathcal{B}^\gamma_{p,q}(\mathbb{R}^2)$.

Usually one needs to introduce weights on the considered spaces. It is occasionally useful to allow for functions (or distributions) that grow even faster then any polynomial. In this case we cannot apply Schwartz's theory of tempered distributions. For this purpose we sometimes work with ultra-distributions, a natural generalization of $\mathcal{S}'(\mathbb{R}^d)$ which allows for faster growing objects. We present this theory in Chapter 2. The discrete analogue will then follow in a natural way and is described in Chapter 3.

Finally, in the treatment of the discrete equations one encounters products of discrete random variables which converge in the limit to ill-defined expression. By now, especially since the work of [DD03], it has become classical to cure such problems by the means of Wick calculus. We present in Section 3.4 a discrete toolbox for this purpose, based on the work of [CSZ17]. For the parabolic Anderson model it is the application of these methods that lets the divergent constant $c^{\varepsilon}_{\mu} \approx \log(\varepsilon^{-1})$ arise.

Chapter 5 & 6: Paraproducts on regularity structures & Schauder theory for singular SPDEs using paraproducts

The key insight in [Hai14] and [GIP15] that lead to the understanding of ill-defined products in singular SPDEs such as $u \cdot \xi$ in 1.9 was that one needs a local description of u in terms of functions (or distributions) constructed from ξ. In the theory of regularity structures [Hai14] this is achieved by providing the concept of a generalized Taylor expansion which allows for non-polynomial terms. More precisely, a distribution on $\mathbb{R}^d$, such as u, is modelled by a function $F : \mathbb{R}^d \to \mathcal{T}$ that takes values in some (abstract) graded vector space

$$\mathcal{T} = \bigoplus_{\alpha \in A} \mathcal{T}_{\alpha},$$

where $A \subseteq \mathbb{R}$ is locally finite. In other words F can be written as $F = \bigoplus_{\alpha \in A} F^{\alpha}$ with only finitely many $F^{\alpha} : \mathbb{R}^d \to \mathcal{T}_{\alpha}$ being not identical with the zero function. This framework is usually equipped with a *model*, a family of linear maps $(\Pi_x, \Gamma_{yx})_{x,y \in \mathbb{R}^d}$, acting like $\Pi_x : \mathcal{T} \to \mathcal{S}'(\mathbb{R}^d)$ and $\Gamma_{yx} : \mathcal{T} \to \mathcal{T}$ and satisfying a number of different properties. The model provides an interpretation that translates the rather formal object $\bigoplus_{\alpha \in A} F^{\alpha}$ into a generalized Taylor expansion with coefficients F^{α}.

$F = \bigoplus_{\alpha < \gamma} F^{\alpha}$ is then said to be a *modelled distribution* of type $\mathcal{D}^{\gamma}(\mathbb{R}^d; \mathcal{T})$ for $\gamma \in \mathbb{R}$ if for any $\alpha \in A$ with $\alpha < \gamma$ and x, $y \in \mathbb{R}^d$

$$F^{\alpha}_{y} - \Gamma^{\alpha}_{yx} F_x = \mathcal{O}(\|y - x\|^{\gamma - \alpha}_{\mathfrak{s}}) \tag{1.13}$$

where $\Gamma^{\alpha}_{yx} F_x$ denotes the component of $\Gamma_{yx} F_x \in \mathcal{T}$ in $\mathcal{T}_{\alpha}$ and where $\|y - x\|_s$ is an "anisotropic" distance induced by some scaling vector $\mathfrak{s}$.

As pointed out above, F is supposed to describe the local expansion of a distribution. In fact, the so-called reconstruction operator $\mathcal{R} : \mathcal{D}^{\gamma}(\mathbb{R}^d; \mathcal{T}) \to \mathcal{S}'(\mathbb{R}^d)$ allows to gain from a modelled distribution the distribution whose local, Taylor-like expansion is provided by F.

It was shown in [GIP15, Theorem 6.10] that the distribution $\mathcal{R}F$ can be "smoothened" via a paraproduct $P(F, \Pi) \in \mathcal{S}'(\mathbb{R}^d)$ in the sense that

$$\mathcal{R}F - P(F, \Pi) \in \mathcal{C}^{\gamma}_{\mathfrak{s}}(\mathbb{R}^d)\,, \tag{1.14}$$

where $\mathcal{C}^{\gamma}_{\mathfrak{s}}(\mathbb{R}^d)$ is the anisotropic Besov space of order γ. We give a definition of these spaces in Chapter 2. It has been conjectured [GIP15, p. 54] that one can describe the space $\mathcal{D}^{\gamma}(\mathbb{R}^d;\mathcal{T})$ completely by using paraproducts.

We here propose to introduce for every $\alpha \in A$ a paraproduct $P(F,\Gamma^{\alpha})$ that smoothens for $F \in \mathcal{D}^{\gamma}(\mathbb{R}^d;\mathcal{T})$ the component F^{α} in the sense

$$F^{\alpha} - P(F,\Gamma^{\alpha}) \in \mathcal{C}^{\gamma-\alpha}_{\mathfrak{s}}(\mathbb{R}^d;\mathcal{T}_{\alpha})\,, \tag{1.15}$$

where the notation $\mathcal{C}^{\gamma-\alpha}_{\mathfrak{s}}(\mathbb{R}^d;\mathcal{T}_{\alpha})$ indicates that the expression actually takes values in the space $\mathcal{T}_{\alpha}$. We show that one has in fact also the inverse direction, namely that (1.15) implies (1.13) given that a *structure condition* is satisfied. To see the need for an extra condition consider the plain vanilla case where F takes values in the so-called polynomial regularity structure $\overline{\mathcal{T}}$. In this case $F \in \mathcal{D}^{\gamma}(\mathbb{R}^d;\overline{\mathcal{T}})$ describes simply an (anisotropic) γ-Hölder continuous function and the components of F are the derivatives of order α. In this simple example it turns out that the paraproducts $P(F,\Gamma^{\alpha})$ are always identical to 0, so that (1.15) describes the right smoothness of these components but says nothing about their interconnections through derivatives. In fact one could in principal choose for F^{α} any $\mathcal{C}^{\gamma-\alpha}$ function and would still satisfy (1.15). This ambivalence can be removed by requiring the *structure condition* on F^{α}, which morally reads as

$$\partial^k(F^{\alpha} - \Gamma^{\alpha}_{\cdot x}F_x)(x) = 0 \tag{1.16}$$

for any $x \in \mathbb{R}^d$ and $k \in \mathbb{N}^d$ with $|k|_{\mathfrak{s}} < \gamma - \alpha$ ($|\cdot|_{\mathfrak{s}}$ denotes an anisotropic multiindex size). In the example above one easily checks that (1.16) yields precisely the right shape of the components of F^{α}.

In general we say that $F : \mathbb{R}^d \to \mathcal{T}$ with $F^{\alpha} = 0$ for $\alpha > \gamma$ is in the paracontrolled space $\mathcal{P}^{\gamma}(\mathbb{R}^d;\mathcal{T})$ if (1.15) and (1.16) are satisfied for $\alpha < \gamma$. In Theorem 5.2.1 we show that indeed

$$\mathcal{D}^{\gamma}(\mathbb{R}^d;\mathcal{T}) = \mathcal{P}^{\gamma}(\mathbb{R}^d;\mathcal{T})\,, \tag{1.17}$$

which proves the conjecture stated in [GIP15].

In Chapter 6 we use (1.17) to prove Schauder estimates for differential operators in the form $a(D) := \partial_t - p(D')$ where $p(D')$ is some homogeneous polynomial of spatial derivatives of order $\theta \in 2\mathbb{N}$. The Green's function of $a(D)$ will be denoted by $\mathscr{A}$. Assume that the considered regularity structure $\mathcal{T}$ contains the polynomial substructure $\overline{\mathcal{T}} \subseteq \mathcal{T}$. As in [Hai14] we then define a model on $\mathcal{T}$ which realizes an abstract integration map $\mathcal{I}$ for the kernel $\mathscr{A}$, but rather decompose $\mathscr{A}$ in its Fourier spectrum than in real space, to exploit the Fourier properties of $\mathscr{A}$. Similar to [GIP15, GP15b] one of the useful properties of the paraproduct is that it commutes with the Green's function $\mathscr{A}$ in the sense that for some $F \in \mathcal{D}^{\gamma}(\mathbb{R}^d;\mathcal{T})$

$$P(\mathcal{I}(F),\Pi) - \mathscr{A} * P(F,\Pi) \in \mathcal{C}^{\gamma+\theta}(\mathbb{R}^d) \tag{1.18}$$

where $P(\cdot,\Pi)$ denotes the paraproduct which we already encountered in 1.14. As in [Hai14] we aim at building a map $\mathcal{K} : \mathcal{D}^\gamma(\mathbb{R}^d,\mathcal{T}) \to \mathcal{D}^{\gamma+\theta}(\mathbb{R}^d;\mathcal{T})$ such that $\mathcal{R}\mathcal{K}F = \mathscr{A} * \mathcal{R}F$. $\mathcal{K}F$ will be chosen as the sum of linear operators

$$\mathcal{K}F = \mathcal{I}F + \mathcal{P}F\,,$$

where $\mathcal{P}F$ takes values in the polynomial substructure $\overline{\mathcal{T}} \subseteq \mathcal{T}$ and $\mathcal{I}$ is the abstract integration map. The estimates on $\mathcal{I}F$ turn out to be rather easy, so that the interesting part is therefore the polynomial contribution $\mathcal{P}F$. We proceed, in contrast to the theory in [Hai14], using paraproducts. One can usually assume [Hai14, Remark 7.9] that (the "relevant" part of) $\mathcal{K}F$ is function-like in the sense of [Hai14, Definition 2.5], which implies that the component of $\mathcal{P}F$ for $\alpha = 0$ must equal $\mathcal{P}^0F = \mathscr{A} * \mathcal{R}F$ [Hai14, Proposition 3.28]. The remaining components of $\mathcal{P}F$ can then be deduced using the structure condition (1.16), which gives an explicit formula for $\mathcal{P}F$ given by (6.31) below. Let us sketch how one proves that $\mathcal{P}^0F$ can be smoothened with the paraproduct $P(\mathcal{K}F,\Gamma^0)$. An important observation is that the paraproduct "forgets" polynomial entries such as $\mathcal{P}F$ so that we have

$$P(\mathcal{K}F,\Gamma^0) = P(\mathcal{I}F,\Gamma^0) \overset{(*)}{=} P(\mathcal{I}F,\Pi)\,,$$

where $(*)$ is a consequence of the fact that (the "relevant" part of) $\mathcal{I}F$ is function-valued. Consequently

$$\begin{aligned}\mathcal{K}^0F - P(\mathcal{K}F,\Gamma^0) &= \mathcal{K}^0F - P(\mathcal{I}F,\Pi) = \mathscr{A} * \mathcal{R}F - P(\mathcal{I}F,\Pi)\\ &\overset{(*_1)}{=} \mathscr{A} * (\mathcal{R}F - P(F,\Pi)) + \mathcal{C}^{\gamma+\theta}_{\mathfrak{s}} \overset{(*_2)}{=} \mathcal{C}^{\gamma+\theta}_{\mathfrak{s}}\,,\end{aligned}$$

where $\mathcal{C}^{\gamma+\theta}_{\mathfrak{s}}$ denotes a term in the corresponding space and where we used in $(*_1)$ that $\mathscr{A}$ commutes with the paraproduct as stated in (1.18) and in $(*_2)$ relation (1.14) together with the fact that $\mathscr{A}$ maps $\mathcal{C}^\gamma$ into $\mathcal{C}^{\gamma+\theta}$. We have thus shown that $\mathcal{K}^0F - P(\mathcal{K}F,\Gamma^0) \in \mathcal{C}^{\gamma+\theta} = \mathcal{C}^{\gamma+\theta-0}$. Using the structure condition (1.16) this result can then be "lifted" to the remaining components of $\mathcal{P}F$ in order to show that $\mathcal{P}^\alpha F - P(F,\Gamma^\alpha) \in \mathcal{C}^{\gamma+\theta-\alpha}$. Applying then (1.17) shows that $\mathcal{K}F \in \mathcal{D}^{\gamma+\theta}(\mathbb{R}^d;\mathcal{T})$ This is in essence the idea behind our Schauder estimates in Theorem 6.2.3, our main result in Chapter 6.

Alas, we simplified the statements to quite some extend. Actually, since we want to consider a parabolic equation we actually only work on a finite time interval $(0,T]$ and moreover must allow for a blowup around the initial condition at $t = 0$. We thus have to work with local, singular spaces. The spaces $\mathcal{D}^{\gamma,\eta}$ introduced in [Hai14] are ill-suited for our purposes since there is no obvious version of 1.17 for them. We choose therefore to introduce a new space $\mathcal{D}^{[\eta,\gamma]}$ in Chapter 5 which roughly behaves in the same way under composition and multiplication and which is better behaved with a Fourier description. In fact $\mathcal{D}^{[\eta,\gamma]}$ is just defined by a collection of estimates in the non-singular spaces $\mathcal{D}^{\beta}$, whose interplay with paraproducts we perfectly understand due to 1.17. Since a Fourier based theory relies on global estimates we have to apply extension operators in order to overcome the locality imposed by the finite time interval $(0,T]$, we introduce two types of extensions in Chapter 5. Let us especially point out the Whitney extension Theorem 5.3.16, which states that any modelled distribution on a closed set can be extended to the full space.

Chapter 7: A regularity structure for rough volatility

In Chapter 7 we give an approximation theory for Itô integrals of the type

$$\int_0^T f(\hat{W}_t)\,\mathrm{d}W_t$$

Here W_t is a Brownian motion and $\hat{W}_t$ is a (Riemann-Liouville) fractional Brownian motion given by

$$\hat{W}_t = \int_0^t K(t-r)\mathrm{d}W_r\,,$$

where $K(r) = \sqrt{2H}\,\mathbf{1}_{r>0}\,r^{H-1/2}$ is the Volterra kernel and $H \in (0,1)$ is the Hurst index. Note that for $H = 1/2$ one has $\hat{W} = W$. Problems of this kind arise in the task for option pricing in financial mathematics. In [BFG16] it is proposed to price a European call option with strike price K by

$$\mathbb{E}\left[C_{\text{B.S.}}\left(S_0\,\exp\left(\int_0^T f(\hat{W}_t)\,\mathrm{d}W_t - \frac{\rho^2}{2}\int_0^T f(\hat{W}_t)\mathrm{d}t\right),\,K,\,\frac{\sqrt{1-\rho^2}}{2}\int_0^T f(\hat{W}_t)^2\mathrm{d}t\right)\right], \tag{1.19}$$

here $C_{\text{B.S.}}(\cdot,\cdot,\cdot)$ denotes the "classical" Black Scholes price, $\rho \in [0,1]$ is some fixed parameter and S_0 is the initial price of the considered asset. We sketch the derivation of this formula in Chapter 7. In (1.19) $f(\hat{W}_t)$ is the volatility process and in pratical applications it is "rough", meaning that $H < 1/2$ and typically $H \sim 0.1$ [GJR17]. Assume we want to approximate (1.19) by taking some (smooth) approximations $W^{\varepsilon} \xrightarrow{\varepsilon\to 0} W$ and $\hat{W}^{\varepsilon}_t = \int_0^t K(t-r)\mathrm{d}W^{\varepsilon}_r$. Does this give us a good approximation? Do we have (and with which rate)

$$\int_0^T f(\hat{W}^{\varepsilon}_t)\,\mathrm{d}W^{\varepsilon}_t \xrightarrow{\varepsilon\to 0} \int_0^T f(\hat{W}_t)\,\mathrm{d}W_t\,? \tag{1.20}$$

We see the connections to the other questions depicted in this introduction by taking the distributional derivative (with respect to T), so that (1.20) can be reformulated as

$$f(\hat{W}_t^\varepsilon)\,\dot{W}_t^\varepsilon \xrightarrow{\varepsilon\to 0} f(\hat{W}_t)\,\dot{W}_t\,, \tag{1.21}$$

where the product on the right hand side is *defined* to be the distributional derivative of the right hand side of (1.20). The problem can from this point of view be understood as the task to find a theory of the distributional product $f(\hat{W}_t)\,\dot{W}_t$ which is robust under approximation. By the results of Wong and Zakai [WZ65] we expect that (1.20) (and thus (1.21)) does *not* hold, but that the sequence $\int_0^T f(\hat{W}_t^\varepsilon)\,\mathrm{d}W_t^\varepsilon$ converges rather to the Stratonovich version of $\int_0^T f(\hat{W}_t)\,\mathrm{d}W_t$. But now one sees that this object does not even exist for $H \in (0, 1/2)$, since the quadratic covariation which yields the Itô-Stratonovich correction diverges in this regime. Note however, that the Itô integral in (1.20) is perfectly well-defined. We expect therefore that one has to subtract some divergent object in (1.20) and (1.21), which somehow corresponds to the infinite Itô-Stratonovich correction. We will apply the theory of regularity structures [Hai14] to achieve this task. From the perspective of this theory one can understand the products in (1.21) by expanding $f(\hat{W}_t^\varepsilon)$ locally, say in the point s close to t,

$$f(\hat{W}_t^\varepsilon) = \sum_{k=0}^{M} f^{(k)}(\hat{W}_s^\varepsilon)\,(\hat{W}_t^\varepsilon - \hat{W}_s^\varepsilon)^k + \mathcal{O}(|s-t|^{(M+1)(H-\kappa)})\,,$$

where we used Taylor's formula and the fact that $\hat{W}$ and thus $\hat{W}^\varepsilon$ is Hölder continuous with exponent $H-\kappa$ for some arbitrarily small $\kappa > 0$. It will be enough to choose M big enough such that $(M+1)(H-\kappa) > 1/2$. Using the reconstruction theorem from regularity structures we can see that this local expansion allows us to get a limit for $f(\hat{W}_t^\varepsilon)\,\dot{W}^\varepsilon$ if we understand the limit of the product of the "monomials" $(\hat{W}_t^\varepsilon - \hat{W}_s)^k$ with $\dot{W}_t^\varepsilon$:

$$(\hat{W}_t^\varepsilon - \hat{W}_s^\varepsilon)^k\,\dot{W}_t^\varepsilon\,. \tag{1.22}$$

Alas, this object is once more divergent as $\varepsilon \to 0$ by the same argument as above. However, we can modify (1.22) to a convergent object, by taking the Wick product instead

$$(\hat{W}_t^\varepsilon - \hat{W}_s^\varepsilon)^k \diamond \dot{W}_t^\varepsilon\,. \tag{1.23}$$

We here assume that $\dot{W}^\varepsilon$ and thus $\hat{W}^\varepsilon$ are Gaussian processes, so that (1.23) should be read as follows: Expand the ordinary product $(\hat{W}_t^\varepsilon - \hat{W}_s^\varepsilon)^k$ in its components in homogeneous Wiener chaoses and then sum the Wick product of each of these terms with $\dot{W}^\varepsilon_t$. This object is now indeed convergent (in the distributional sense) to a limit which turns out (for $s < t$) to be $(\hat{W}_t - \hat{W}_s)\,\dot{W}_t := \big(\int_s^\cdot (\hat{W}_r - \hat{W}_s)\mathrm{d}W_r\big)'(t)$. Using Wick calculus one can see that (again $s < t$ for simplicity)

$$(\hat{W}^\varepsilon_t - \hat{W}^\varepsilon_s)^k \diamond \dot{W}^\varepsilon_t = (\hat{W}^\varepsilon_t - \hat{W}^\varepsilon_s)^k\,\dot{W}^\varepsilon_t - k\mathscr{C}^\varepsilon(t)\,(\hat{W}^\varepsilon_t - \hat{W}^\varepsilon_s)^{k-1}\,,$$

where $\mathscr{C}^\varepsilon(t) = \mathbb{E}[\hat{W}^\varepsilon_t \dot{W}^\varepsilon_t] \sim \varepsilon^{H-1/2}$. Note that the correction $k\mathscr{C}^\varepsilon(t)\,(\hat{W}^\varepsilon_t - \hat{W}^\varepsilon_s)^{k-1}$ does only have non-positive homogeneity for $k = 1$, by what we mean that for $k > 1$ this term can be bounded by $|t-s|^{\kappa'}$ for some $\kappa' > 0$. Consequently, after "sewing" everything together with the reconstruction theorem [Hai14, Theorem 3.10], only the $k = 1$ correction survives and one obtains the following modified convergence result

$$\int_0^T f(\hat{W}^\varepsilon_t)\,\mathrm{d}W^\varepsilon_t - \int_0^T \mathscr{C}^\varepsilon(t)\,f'(\hat{W}^\varepsilon_t)\,\mathrm{d}t \overset{\varepsilon\to 0}{\longrightarrow} \int_0^T f(\hat{W}_t)\,\mathrm{d}W_t\,. \tag{1.24}$$

This shoud be seen as the correct version of the wrong statement (1.21). The fundamental objects (1.22) and its "renormalization" (1.23) are realized in Chapter 7 by a model Π^ε and a renormalized model $\hat{\Pi}^\varepsilon$. The limiting building blocks $(\hat{W}_t - \hat{W}_s)\,\dot{W}_t$ are represented by a limiting model $\hat{\Pi}$. In Theorem 7.1.13 we show that $\hat{\Pi}^\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} \hat{\Pi}$ and thereafter apply this knowledge together with the reconstruction theorem to deduce (1.24) (together with a convergence rate) in our main Theorem 7.2.9.

As we do not necessarily construct our approximation W^ε through convolution with a mollifier, the "renormalization function" $\mathscr{C}^\varepsilon(t)$ is in general really a function and not a constant, which is the case one usually encounters for singular SPDEs as in Chapter 4. For the class of used approximations W^ε see (7.16) together with Definition 7.1.5.

Many of the ideas above seem quite reminiscent to rough path theory. In fact it would probably be possible to produce the results above with the branched rough path theory as introduced by Gubinelli [Gub10]. However, note that a full branched rough path framework is not possible since integrals such as $\int_s^t (W_r - W_s)\mathrm{d}\hat{W}_r$ do not even exists as an Itô integral because $\hat{W}$ is not a semimartingale. Most likely the theory above could instead be achieved with a "partial" branched rough path framework, but we find it more straightforward and economic to apply Hairer's language of regularity structure instead, not at least since we encouter renormalization terms (although the concept of renormalization was translated recently to the theory of rough paths, compare [BCFP17]).

Chapter 8: Solution to (SNLS) on $\mathbb{R}^2$

Recall that the stochastic nonlinear Schrödinger equation on $[0,T] \times \mathbb{R}^2$ is given by

$$(\imath\partial_t - \Delta)u = \lambda|u|^{2\sigma}u + u\xi,\ u(0) = u_0, \tag{1.25}$$

where $\xi \in \mathcal{S}'(\mathbb{R}^2)$ is white noise in space and where we take $\sigma > 0$ and $\lambda \in \mathbb{R}$. This equation models light propagation in a dispersive material (represented by the nonlinear term $\lambda|u|^{2\sigma}u$) [Ber98, Section 1.1.-1.3.], the multiplicative noise term can be seen as an attempt to take impurities in the material into account. The solvability of this equations is sensitive to the choice of the "material parameters" σ and λ. The case $\sigma < 1$ is known as the *subcritical* regime. The sign of λ classifies (1.25) as *focusing* ($\lambda > 0$), *linear* ($\lambda = 0$) or *defocusing* ($\lambda \leqslant 0$).

Again, as for the parabolic Anderson model, the product term $u \cdot \xi$ is ill-defined as a product of a (non-smooth) function with white noise. Since we don't have any information about u a priori it is hard to say how we could define a product. The techniques from [GIP15] and [Hai14] are unavailable for us, since they rely on smoothing properties which are not available for the semigroup $e^{-\imath t\Delta}$. One only has, by Parseval's identity, that $e^{-\imath t\Delta}$ maps $L^2(\mathbb{R}^2)$ into itself, compare this to the heat semigroup $e^{t\Delta}$ that maps $L^2(\mathbb{R}^2)$ (or any tempered distribution) into the class of analytical functions!

In a nutshell, the idea to overcome this obstacle is to transform (1.25) into a better behaved equation and and then to study the regularity of the solution via conserved quantities. We follow [HL15, DW16] and transform (1.25) by a "partial" Cole Hopf transform: Consider instead of u the function $v = ue^Y$, where Y is a *time-independent* function solving $\Delta Y = \xi$. Instead of (1.25) we then have

$$(\imath\partial_t - \Delta)v = v\,\nabla Y^2 - 2\nabla v \cdot \nabla Y + \lambda|v|^{2\sigma}ve^{-2\sigma Y},\ v(0) = v_0 := u_0 e^Y$$

The derivative ∇Y is once more a distribution, so that the square $\nabla Y^2 = \nabla Y \cdot \nabla Y$ is ill-defined. However, since we understand Y much better than the solution of the equation, we can fix this problem in a rather easy way. Similar as in [DD03] we replace this equation by

$$(\imath\partial_t - \Delta)v = v\ \nabla Y^{\diamond 2} - 2\nabla v \cdot \nabla Y + \lambda|v|^{2\sigma}ve^{-2\sigma Y},\ v(0) = v_0\,, \tag{1.26}$$

where $\nabla Y^{\diamond 2}$ denotes the Wick product $\nabla Y \diamond \nabla Y$. If we look at the equation for $u = ve^{-Y}$ the replacement $\nabla Y^2 \to \nabla Y^{\diamond 2}$ corresponds to the renormalization seen in Chapter 4 for the parabolic Anderson model. The scalar product $-2\nabla v \cdot \nabla Y$ is once more ill-defined, but by the rules of classical analysis [BCD11, Theorem 2.85] it can be defined if we can show that for $t \in [0,T]$ the function $v(t) = v(t,\cdot) : \mathbb{R}^2 \to \mathbb{C}$ is contained in the fractional Sobolev space $H^\gamma(\mathbb{R}^2)$ with $\gamma > 1$. This is achieved by working with conserved quantities of this equation. For our proceeding this will be the mass N and the energy H, given by

$$N = \int_{\mathbb{R}^2} |v|^2 e^{-2Y},\ H = \int_{\mathbb{R}^2} \big(\frac{1}{2}|\nabla v|^2 - \frac{1}{2}|v|^2 \nabla Y^{\diamond 2} - \frac{\lambda}{2\sigma+2}|v|^{2\sigma+2}e^{-2\sigma Y}\big)e^{-2Y} \tag{1.27}$$

From these one can derive an $H^1(\mathbb{R}^2)$ bound if the equation is defocusing or subcritical (or both). The most challenging term in this proceeding is the nonlinear contribution $\int \frac{\lambda}{2\sigma+2}|v|^{2\sigma+2}e^{-(2\sigma+2)Y}$ to the energy. If $\lambda \leqslant 0$ this contribution can be neglected in the estimates, if $\sigma \in (0,1)$ one can bound this integral by using Sobolev embedding.

An estimate in $H^2(\mathbb{R}^2)$, or rather $H^\gamma(\mathbb{R}^2)$ for $\gamma \in (1,2)$, follows by bounding the object $w = \partial_t v$ in $L^2(\mathbb{R}^2)$ and applying thereafter the identity $\frac{1}{\imath}\Delta v + \ldots = w$. This is the same path that was chosen in [DW16] to solve (1.26) (globally) on the torus $\mathbb{T}^2$. In this case one can estimate terms derived from the noise such as $\nabla Y^{\diamond 2}$, ∇Y and e^{-2Y} without the introduction of weights.

We here prove that (1.26) does have local solutions on all of $\mathbb{R}^2$ if the equation is defocusing or subcritical (or both). If $\sigma \in (0,1/2)$ one even has global solutions. The difficulty of a set-up on $\mathbb{R}^2$ is that one has to counterbalance the growth of the noise with some weight, compare also in this context Chapter 4. Consequently, we have to show that the solution v has a certain decay, since otherwise the conserved quantities (1.27) are not even well-defined. Luckily, the growth of ξ and Y is about $\sqrt{\log(|x|)}$, so that e^{-2Y}, ∇Y and $\nabla Y^{\diamond 2}$ grow less than $|x|^\kappa$ for any $\kappa > 0$. To compensate this behaviour we have to show that the solution v of (1.26) decays faster than some polynomial. The key role in this task will be played by Lemma 8.3.1, whose statement can be summarized for $\delta \in (0,1/2)$, $\delta' < 1-2\delta$ as

$$\||x|^\delta v\|^2_{C([0,T];L^2(\mathbb{R}^2))} = \sup_{t\in[0,T]} \||x|^\delta v(t)\|^2_{L^2(\mathbb{R}^2)} \lesssim 1 + \||x|^{-\delta'}\nabla v(t)\|_{C([0,T];L^2(\mathbb{R}^2))}$$

We use this lemma together with interpolation of weighted Besov spaces (Lemma 2.1.31) to “trade” some differentiability to gain some a priori decay estimates for v.

Let us finally sketch the role of the parameter $\sigma > 0$ in proving existence of local or global solutions. We simplify the appearing estimates greatly to ease the argument. When estimating the H^γ norm for $\gamma > 1$ we get in Lemma 8.4.1 an estimate which is roughly of the form

$$\|v\|_{C([0,T];H^\gamma)} \lesssim 1 + e^{CT\|v\|^{2\sigma}_{C([0,T];L^\infty)}} \tag{1.28}$$

To prove local existence one can use the embedding $\|v\|_{C([0,T];L^\infty)} \lesssim \|v\|_{C([0,T];H^\gamma)}$ and then choose $T > 0$ small enough to get from (1.28) an a priori bound on $\|v\|_{C([0,T];H^\gamma)}$.To prove global existence one can use the Brezis-Gallouet like inequality

(Lemma 8.5.1) $\|v\|_{C([0,T];L^\infty)} \lesssim 1 + \log(1 + \|v\|_{C([0,T];H^\gamma)})$ which gives for $\sigma < 1/2$ a sublinear bound on the right hand side of (1.28) and thus a global estimate. Using these a priori bounds one derives the existence of local solutions and of global solutions (for $\sigma \in (0, 1/2)$), these main results are stated in Theorem 8.4.4 and 8.5.2.

1.1 Notation

The most important symbols (and some important terms) can be found in the glossary at the end of this thesis. We list here a few important notions and notations.

Constants and Inequalities

For expressions a, b we introduce the notation

$$a \lesssim b \qquad \Leftrightarrow \qquad a \leqslant C \cdot b\,,$$

where C is some *deterministic* constant, independent of a and b. To emphasize the dependency of C on some parameter p a notation such as "$\lesssim_p$" will be used. We further introduce

$$\begin{aligned} a \gtrsim b \qquad &\Leftrightarrow \qquad b \lesssim a\,, \\ a \approx b \qquad &\Leftrightarrow \qquad a \lesssim b \,\&\, b \lesssim a\,. \end{aligned}$$

For indices $i, j \in \mathbb{Z}$ we will write

$$i \precsim j \qquad \Leftrightarrow i \leqslant j + N \tag{1.29}$$

where $N \in \mathbb{Z}$ is some deterministic constant, which is independent of i and j. We also use

$$\begin{aligned} i \succsim j \qquad &\Leftrightarrow \qquad i \precsim j\,, \\ i \sim j \qquad &\Leftrightarrow \qquad i \precsim j \,\&\, j \precsim i\,. \end{aligned}$$

Multi-indices

We write in this thesis $\mathbb{N}$ for the natural numbers including 0.

For multi-indices $k,\, l \in \mathbb{N}^d$ we introduce the following ordering

$$k \leqslant l \qquad \Leftrightarrow \qquad \forall i \in \{1, \ldots, d\}\;\; k_i \leqslant l_i\,.$$

We further use the usual notations concerning factorials and derivatives:

$$
\begin{aligned}
k! &:= k_1! \cdot \ldots \cdot k_d!\,, \\
|k| &:= k_1 + \ldots + k_d\,, \\
\binom{k}{l} &:= \frac{k!}{l!(k-l)!}\,, \\
\partial^k &:= \frac{\partial^{|k|}}{\partial_{x_1}^{k_1} \ldots \partial_{x_d}^{k_d}}\,.
\end{aligned}
$$

Functions

For functions we will sometimes write the argument as an index to shorten the notation:

$$f_x := f(x)\,.$$

Given an open set $\Omega \subseteq \mathbb{R}^d$ and $n \in \mathbb{N}$ we write

$$f \in C^n(\Omega)$$

whenever $f : \Omega \to \mathbb{C}$ has derivatives in Ω up to order n. As an extension of the notation above we define for $k \in \mathbb{N}^d$

$$\partial^k f_x := (\partial^k f)_x = (\partial^k f)(x)\,.$$

We use the symbol $C_b^n(\Omega)$ for the set of functions in $C^n(\Omega)$ with bounded derivatives on Ω. We may write

$$
\begin{aligned}
C(\Omega) &:= C^0(\Omega) \\
C_b(\Omega) &:= C_b^0(\Omega)
\end{aligned}
$$

for the space of (bounded) continuous functions. Let us also introduce

$$
\begin{aligned}
C^\infty(\Omega) &:= \bigcap_{n \geqslant 0} C^n(\Omega) \\
C_b^\infty(\Omega) &:= \bigcap_{n \geqslant 0} C_b^n(\Omega)
\end{aligned}
$$

The index "c" stands for *compact support*, so that

$$C_c^n(\Omega),\ C_c^\infty(\Omega)$$

will denote functions in $C^n(\Omega)$ or $C^\infty(\Omega)$ with compact support $\operatorname{supp} f \subseteq \Omega$.

We will occasionally use the term *spectral support* to denote the support of the Fourier transform, i.e.

$$\operatorname{supp} \mathcal{F}_{\mathbb{R}^d} f$$

(a similar remark holds for tempered (ultra-)distributions).

When we are considering function spaces on a domain Ω of the form $M(\Omega)$, such as $C^n(\Omega)$, we often do not emphasize the codomain of the considered functions. This should then always be read as indicating that the considered functions take values in the complex numbers $\mathbb{C}$. If we want to emphasize the target set we will use a *semicolon*, so that

$$M(\Omega; X)$$

would be a set of functions for values in X. For example, we might use $C_b(\Omega; \mathbb{R})$ for the space of continuous real-valued functions and use

$$C^n(\Omega; X)$$

with the Banach space X as the symbol for the set of functions that take values in the space X and are n times differentiable.

Important exceptions of this convention will be the spaces $\mathcal{D}^\gamma(\mathbb{R}^d; \mathcal{V}\backslash\mathcal{W})$ and $\mathcal{D}^{[\eta,\gamma]}(\mathbb{R}^d; \mathcal{V}\backslash\mathcal{W})$ we introduce in Definition 2.3.14 and 5.3.1 below. Here the considered functions take values in a *superset* of $\mathcal{V}\backslash\mathcal{W}$, but $\mathcal{V}\backslash\mathcal{W}$ denotes the set where the semi-norm of these spaces is taken, compare the Remarks 2.3.15 and 5.3.2 below.

Products

We will denote by $\cdot$ the "usual product" of real or complex numbers. If the factors of some product span over multiple lines, the symbol $\times$ will be used.

The symbol $\bullet$ is reserved to denote for $a, b \in \mathbb{C}^d$

$$a \bullet b := \sum_{i=1}^{d} a_i \cdot b_i$$

and is thus identical with the usual, euclidean scalar product whenever a, b have real components.

Integrals

For integrals we will often use the "physics notation" where the differential is written right after the integral sign:

$$\int_{\mathbb{R}^d} \mathrm{d}x \, f(x) \,. \tag{1.30}$$

This notation often leads to more well-arranged formulas, especially when multiple integrals occur. If the integration domain is the "full space" we may skip the index on the integral sign, so that we would write instead of (1.30)

$$\int \mathrm{d}x\, f(x)\,.$$

A convolution on $\mathbb{R}^d$ will be denoted by

$$(f * g)(x) := \int_{\mathbb{R}^d} \mathrm{d}z\, f(x-z)\, g(z) = \int_{\mathbb{R}^d} \mathrm{d}z\, f(z)\, g(x-z)\,.$$

We may occassionaly write $*_{\mathbb{R}^d}$ to emphasize that the convolution is really taken on $\mathbb{R}^d$. Convolution over other domains, such as $\mathbb{Z}$ for instance, will be marked by a special label.

The Fourier integral will always follow in this thesis the "frequency" convention, that is

$$\mathcal{F}_{\mathbb{R}^d} f(x) = \int_{\mathbb{R}^d} \mathrm{d}\xi\, e^{-2\pi\imath\xi\cdot x} f(\xi),\ \mathcal{F}^{-1}_{\mathbb{R}^d} f(x) \int_{\mathbb{R}^d} \mathrm{d}\xi\, e^{2\pi\imath\xi\cdot x} f(\xi)\,.$$

For a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ acting on some test function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ we will occasionally apply the "measure notation":

$$\int f(\mathrm{d}z)\, \varphi(z) := f(\varphi)\,.$$

A similar remark applies to ultra-distributions.

Chapter 2

Background

2.1 A recap on Fourier analysis

(Ultra) Distributions

Most of the theory presented in this thesis is based on distributions. Since the paracontrolled analysis of SPDEs as presented in [GIP15] is heavily based on Fourier analysis it seems convenient to work in the framework of Schwartz distributions. Let us shortly recall that this theory is based on the so-called Schwartz space

$$\mathcal{S}(\mathbb{R}^d) := \{\varphi \in C^\infty(\mathbb{R}^d) \,|\, p_{\alpha,\lambda}(\varphi) < \infty\} \tag{2.1}$$

which is equipped with the (semi-)norms

$$p_{\alpha,\lambda}(\varphi) := \sup_{x\in\mathbb{R}^d} |(1+|x|^2)^{\lambda/2}\partial^\alpha \varphi(x)|$$

to make it a locally convex space. Its dual $\mathcal{S}'(\mathbb{R}^d)$ is called the space of tempered distributions. Via the identification

$$T_f(\varphi) = \int f(x)\varphi(x)\mathrm{d}x\,,$$

for say $f \in L^p$, $p \in [1,\infty]$, $S'(\mathbb{R}^d)$ can be seen as a generalization of functions. Objects $T \in \mathcal{S}'(\mathbb{R}^d)$ can be added and multiplied by constants in the obvious way, but also "multiplied" by $\varphi \in \mathcal{S}(\mathbb{R}^d)$ via

$$\varphi T := T(\varphi\,\cdot)$$

They form a subset of the realm of distributions $\left(C_c^\infty(\mathbb{R}^d)\right)'$ with the important additional property that a Fourier transform can be defined via

$$\mathcal{F}_{\mathbb{R}^d}T(\varphi) := T(\mathcal{F}_{\mathbb{R}^d}\varphi),\ \varphi \in \mathcal{S}(\mathbb{R}^d)\,.$$

By definition, $S'(\mathbb{R}^d)$ contains objects that grow at most like a polynomial, in the sense

$$|T(\varphi(\cdot - x))| \lesssim 1 + |x|^N$$

for some $N > 0$. However, as we will see in chapter 4 below it will be sometime convenient to work with objects that grow (or decay) faster than any polynomial. This theory of so-called tempered ultra-distributions was introduced by Beurling and Björck [Beu38, Bjö66]. These objects keep most of the nice properties of Schwartz distributions while allowing to grow at any subexponential rate. In this subsection we present this theory and define fundamendal objects such as Littlewood-Paley decomposition and Besov spaces in their context. We follow here mostly [MP17], which is based on [Tri83].

Let us fix, once and for all, the following weight functions which we will use throughout this thesis.

Definition 2.1.1. *In this thesis we denote by*

$$\omega^{\mathrm{pol}}(x) := \log(1 + |x|),\ \omega^{\exp}_\sigma(x) := |x|^\sigma,\ \sigma \in (0,1)\,,$$

where $x \in \mathbb{R}^d$, $\sigma \in (0,1)$ *For* $\omega \in \boldsymbol{\omega} := \{\omega^{\mathrm{pol}}\} \cup \{\omega^{\exp}_\sigma \,|\, \sigma \in (0,1)\}$ *we denote by* $\boldsymbol{\rho}(\omega)$ *the set of measurable, strictly positive* $\rho : \mathbb{R}^d \to (0,\infty)$ *such that*

$$\rho(x) \lesssim \rho(y) e^{\lambda\omega(x)} \tag{2.2}$$

for some $\lambda = \lambda(\rho) > 0$. *We also introduce the notation* $\boldsymbol{\rho}(\boldsymbol{\omega}) := \bigcup_{\omega\in\boldsymbol{\omega}} \boldsymbol{\rho}(\omega)$. *The objects* $\rho \in \boldsymbol{\rho}(\boldsymbol{\omega})$ *will be called* weights.

Note that the sets $\boldsymbol{\rho}(\omega)$ are stable under addition and multiplication for a fixed $\omega \in \boldsymbol{\omega}$. The indices "pol" and "exp" of the elements in $\boldsymbol{\omega}$ indicate the fact that elements in $\rho \in \boldsymbol{\rho}(\omega^{\mathrm{pol}})$ are polynomially growing or decaying while elements in $\boldsymbol{\rho}(\omega^{\exp}_\sigma)$ are allowed to have subexponential behaviour. Note that

$$\boldsymbol{\rho}(\omega^{\mathrm{pol}}) \subseteq \boldsymbol{\rho}(\omega^{\exp}_\sigma)$$

and that

$$\langle x\rangle^\lambda := (1 + |x|^2)^{\lambda/2} \in \boldsymbol{\rho}(\omega^{\mathrm{pol}}) \tag{2.3}$$

and $e^{\lambda|x|^\sigma} \in \boldsymbol{\rho}(\omega^{\exp}_\sigma)$ for $\lambda \in \mathbb{R}$, $\sigma \in (0,1)$. The reason why we only allow for $\sigma < 1$ will be explained in Remark 2.1.3 below.

We are now ready to define the space of ultra-distributions

Definition 2.1.2. *We define for $\omega \in \boldsymbol{\omega}$ the locally convex space*

$$\mathcal{S}_\omega(\mathbb{R}^d) := \{f \in \mathcal{S}(\mathbb{R}^d) \,|\, \forall \lambda > 0,\, \alpha \in \mathbb{N}^d \qquad p^\omega_{\alpha,\lambda}(f),\, \pi^\omega_{\alpha,\lambda}(f) < \infty\}\,, \tag{2.4}$$

which is equipped with the seminorms

$$p^\omega_{\alpha,\lambda}(f) := \sup_{x\in\mathbb{R}^d} e^{\lambda\omega(x)}|\partial^\alpha f(x)|\,, \tag{2.5}$$

$$\pi^\omega_{\alpha,\lambda}(f) := \sup_{x\in\mathbb{R}^d} e^{\lambda\omega(x)}|\partial^\alpha \mathcal{F}_{\mathbb{R}^d} f(x)|\,. \tag{2.6}$$

Its topological dual $\mathcal{S}'_\omega(\mathbb{R}^d)$ (equipped with the strong topology) is called the space of tempered ultra-distributions.

Remark 2.1.3. *The reason why we excluded the case $\sigma \geqslant 1$ for $\omega^{\exp}_\sigma$ in Definition 2.1.1 is that we want $\mathcal{S}_\omega$ to contain functions with compact support, which then allows for localization and thus for a Littlewood-Paley theory. But if $\omega = \omega^{\exp}_\sigma$ with $\sigma \geqslant 1$ and $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ the requirement $\pi^\omega_{0,\lambda}(f) < \infty$ implies that $\mathcal{F}_{\mathbb{R}^d} f$ can be bounded by $e^{-c|x|}$, $c > 0$, which means that f is analytic and the only compactly supported $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ is the zero-function $f = 0$.*

In the case $\omega = \omega^{\exp}_\sigma, \sigma \in (0,1)$ the space $\mathcal{S}'_\omega$ is strictly larger than $\mathcal{S}'$. Indeed: $e^{c|\cdot|^{\sigma'}} \in \mathcal{S}'_\omega \backslash \mathcal{S}'$ for $\sigma' \in (0,\sigma]$. In the case $\omega = \omega^{\mathrm{pol}}$ we simply have

$$\mathcal{S}_\omega(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d)$$

with a topology that can also be generated by only using the seminorms $p^\omega_{\alpha,\lambda}$ so that the dual of $\mathcal{S}_\omega(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d)$ is given by

$$\mathcal{S}'_\omega(\mathbb{R}^d) = \mathcal{S}'(\mathbb{R}^d)\,.$$

The theory of “classical” tempered distributions is therefore contained in the framework above.

The role of the triple

$$C^\infty_c(\mathbb{R}^d) \subseteq \mathcal{S}(\mathbb{R}^d) \subseteq C^\infty(\mathbb{R}^d)$$

in this theory will be substituted by spaces $C^\infty_{\omega,c}(\mathbb{R}^d)$, $C^\infty_\omega(\mathbb{R}^d)$ such that

$$C^\infty_{\omega,c}(\mathbb{R}^d) \subseteq \mathcal{S}_\omega(\mathbb{R}^d) \subseteq C^\infty_\omega(\mathbb{R}^d)\,.$$

Definition 2.1.4. *Let $U \subseteq \mathbb{R}^d$ be an open set and $\omega \in \boldsymbol{\omega} = \{\omega^{\mathrm{pol}}\} \cup \{\omega_\sigma^{\exp} \mid \sigma \in (0,1)\}$. We define for $\omega = \omega_\sigma^{\exp}$ the set $C_\omega^\infty(U)$ to be the space of $f \in C^\infty(U)$ such that for every $\varepsilon > 0$ and compact $K \subseteq U$ there exists $C_{\varepsilon,K} > 0$ such that for all $\alpha \in \mathbb{N}^d$*

$$\sup_K |\partial^\alpha f| \leqslant C_{\varepsilon,K}\, \varepsilon^{|\alpha|} (\alpha!)^{1/\sigma} . \tag{2.7}$$

For $\omega = \omega^{\mathrm{pol}}$ we set $C_\omega^\infty(U) = C^\infty(U)$. We also define

$$C_{\omega,c}^\infty(U) = C_\omega^\infty(U) \cap C_c^\infty(U) . \tag{2.8}$$

The elements of $\big(C_{\omega,c}^\infty(\mathbb{R}^d)\big)'$ are called ultra-distributions.

Remark 2.1.5. *The space $\big(C_{\omega,c}^\infty(\mathbb{R}^d)\big)'$ is equipped with some suitable topology [Bjö66, Section 1.6] which we did not specify since this space will not be used in this thesis and is just mentioned for the sake of completeness.*

Remark 2.1.6. *The factor $\alpha!$ in (2.7) can be replaced by $|\alpha|!$ or $|\alpha|^{|\alpha|}$ [Rod93, Proposition 1.4.2] as can be easily seen from $\alpha! \leqslant |\alpha|! \leqslant d^{|\alpha|}\alpha!$ and Stirlings formula.*

The relation between $C_{\omega,c}^\infty, \mathcal{S}_\omega, C_\omega^\infty$ and their properties are specified by the following lemma.

Lemma 2.1.7. *For $\omega \in \boldsymbol{\omega}$ we have $\mathcal{S}_\omega(\mathbb{R}^d) \subseteq C_\omega^\infty(\mathbb{R}^d)$ and*

$$C_{\omega,c}^\infty(\mathbb{R}^d) = \mathcal{S}_\omega(\mathbb{R}^d) \cap C_c^\infty(\mathbb{R}^d) . \tag{2.9}$$

In particular $C_{\omega,c}^\infty(\mathbb{R}^d) \subseteq \mathcal{S}_\omega(\mathbb{R}^d) \subseteq C_c^\infty(\mathbb{R}^d)$.

The space $\mathcal{S}(\mathbb{R}^d)$ is stable under addition, multiplication and convolution.

The space $C_\omega^\infty(\mathbb{R}^d)$ is stable under addition, multiplication and division in the sense that $f/g \cdot \mathbf{1}_{\operatorname{supp} f} \in C_\omega^\infty(\mathbb{R}^d)$ for $f, g \in C_\omega^\infty(\mathbb{R}^d)$, $\operatorname{supp} f \subseteq \overset{\circ}{\operatorname{supp}}\, g$.

Sketch of the proof. We only have to prove the statements for $\omega \in \{\omega_\sigma^{\exp} \mid \sigma \in (0,1)\}$. Take a $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ and $\varepsilon > 0$. We then have for $\alpha \in \mathbb{N}^d$

$$\partial^\alpha f(x) = (2\pi\imath)^{|\alpha|} \int_{\mathbb{R}^d} e^{2\pi\imath x\cdot\xi}\, \xi^\alpha\, \mathcal{F}_{\mathbb{R}^d} f(\xi)\, \mathrm{d}\xi$$

Using further that for $\lambda > 0$ (we here follow [Hör05, Lemma 12.7.4.])

$$\begin{aligned}\int |\xi|^{|\alpha|} e^{-\lambda|\xi|^\sigma} \mathrm{d}\xi &\lesssim \int_0^\infty r^{|\alpha|+d-1} e^{-\lambda r^\sigma} \mathrm{d}r \lesssim \lambda^{-|\alpha|/\sigma} \Gamma\big((|\alpha|+d)/\sigma\big) \\ &\overset{\text{Stirling}}{\lesssim} \lambda^{-|\alpha|/\sigma} C^{|\alpha|} |\alpha|^{|\alpha|/\sigma}\end{aligned}$$

we obtain for $x \in \mathbb{R}^d$

$$|\partial^\alpha f(x)| \lesssim C_\lambda \lambda^{-|\alpha|/\sigma} C^{|\alpha|} |\alpha|^{|\alpha|/\sigma} \cdot \pi^\omega_{0,\lambda}(f)\,.$$

Choosing $\lambda > 0$ big enough shows (2.7) (with global bounds) and thus $f \in C^\infty_\omega(\mathbb{R}^d)$ and $\mathcal{S}_\omega(\mathbb{R}^d) \subseteq C^\infty_\omega(\mathbb{R}^d)$. In particular we get $\mathcal{S}_\omega(\mathbb{R}^d) \cap C^\infty_c(\mathbb{R}^d) \subseteq C^\infty_{\omega,c}(\mathbb{R}^d)$. To show the inverse inclusion consider $f \in C^\infty_{\omega,c}(\mathbb{R}^d)$. We only have to show that for any $\lambda > 0$ and $\alpha \in \mathbb{N}^d$ $\pi^\omega_{\alpha,\lambda}(f) < \infty$. And indeed for $x \in \mathbb{R}^d$ with (without loss of generality) $|x| \geqslant 1$ [1]

$$\begin{aligned}
|e^{\lambda|x|^\sigma} \mathcal{F}_{\mathbb{R}^d} f(x)| &\leqslant \sum_{k=0}^\infty \frac{\lambda^k}{k!} |x|^{\sigma k} |\mathcal{F}_{\mathbb{R}^d} f(x)| \leqslant \sum_{k=0}^\infty \frac{\lambda^k C^k}{k!} |x|^{\lceil \sigma k \rceil} |\mathcal{F}_{\mathbb{R}^d} f(x)| \\
&\leqslant \sum_{i=1}^d \sum_{k=0}^\infty \frac{\lambda^k C^k}{k!} |x_i|^{\lceil \sigma k \rceil} |\mathcal{F}_{\mathbb{R}^d} f(x)| = \sum_{i=1}^d \sum_{k=0}^\infty \frac{\lambda^k C^k}{k!} \Big| \int e^{2\pi \imath \xi} \partial^{\lceil \sigma k \rceil e_i} f(\xi)\, \mathrm{d}\xi \Big| \\
&\overset{2.7\,\&\,\text{Stirling}}{\leqslant} C_\varepsilon \sum_{k=0}^\infty \lambda^k C^k \varepsilon^k < \infty
\end{aligned}$$

where $C, C_\varepsilon > 0$ denote as usual constants that may change from line to line and where $\varepsilon > 0$ was chosen small enough in the last step.

The stability of $\mathcal{S}_\omega(\mathbb{R}^d)$ under addition, multiplication and convolution are quite easy to check. [Bjö66, Proposition 1.8.3].

It is straightforward to check $f \cdot g \in C^\infty_\omega(U)$ for $f, g \in C^\infty_\omega(U)$ using Leibniz's rule. For the stability under composition see e.g. [RS12, Proposition 3.1], from which the stability under division can be easily derived □

Many linear operations such as addition or derivation that can be defined on distributions can be translated immediately to the space of ultra-distributions $\big(C^\infty_{\omega,c}(\mathbb{R}^d)\big)'$. We see with (2.8) that $C^\infty_\omega(\mathbb{R}^d)$ should be interpreted as the set of permitted smooth multipliers for ultra-distributions $\big(C^\infty_{\omega,c}(\mathbb{R}^d)\big)'$ and in particular for tempered ultra-distributions $\mathcal{S}'_\omega(\mathbb{R}^d) \subseteq \big(C^\infty_{\omega,c}(\mathbb{R}^d)\big)'$.

The space $\mathcal{S}'_\omega(\mathbb{R}^d)$ is small enough to allow for a Fourier transform.

Definition 2.1.8. *For $f \in \mathcal{S}'_\omega(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}_\omega(\mathbb{R}^d)$ we set*

$$\begin{aligned}
\mathcal{F}_{\mathbb{R}^d} f(\varphi) &:= \widehat{f}(\varphi) := f(\mathcal{F}_{\mathbb{R}^d}\varphi), \\
\mathcal{F}^{-1}_{\mathbb{R}^d} f(\varphi) &:= \check{f}(\varphi) := f(\mathcal{F}^{-1}_{\mathbb{R}^d}\varphi).
\end{aligned}$$

By definition of $\mathcal{S}_\omega(\mathbb{R}^d)$ we have that $\mathcal{F}_{\mathbb{R}^d}$ and $\mathcal{F}^{-1}_{\mathbb{R}^d}$ are isormophisms on $\mathcal{S}_\omega(\mathbb{R}^d)$ which implies that $\mathcal{F}_{\mathbb{R}^d}$ and $\mathcal{F}^{-1}_{\mathbb{R}^d}$ are isomorphisms on $\mathcal{S}'_\omega(\mathbb{R}^d)$.

[1] we here follow in principle ideas from [MW15, Proposition A.2].

The following lemma proves that the set of compactly supported ultra-differentiable functions $C^\infty_{\omega,c}(\mathbb{R}^d)$ is rich enough to localize ultra-distributions, which gets the Littlewood-Paley theory started and allows us to introduce Besov spaces based on ultra-distributions in the next section.

Lemma 2.1.9 ([Bjö66], Theorem 1.3.7.)**.** *Let $\omega \in \boldsymbol{\omega}$. For every pair of compact sets $K \subsetneq K' \subseteq \mathbb{R}^d$ there is a $\varphi \in C^\infty_{\omega,c}(\mathbb{R}^d)$ such that*

$$\varphi|_K = 1 , \qquad \operatorname{supp} \varphi \subseteq K' .$$

Weighted, anisotropic Besov spaces

We will use the following convention for weighted L^p spaces: For $\rho \in \boldsymbol{\rho}(\omega)$ we write

$$\|f\|_{L^p(\mathbb{R}^d,\rho)} := \|\rho \cdot f\|_{L^p(\mathbb{R}^d)} \tag{2.10}$$

and write $L^p(\mathbb{R}^d, \rho)$ for the space of all measurable f for which this is finite.

For the function spaces we want to define next some more work is needed. Let us first introduce some *scaling vector*

$$\mathfrak{s} \in [1, \infty)^d , \tag{2.11}$$

which we fix throughout this subsection. We further set, as in [Hai14], for $x \in \mathbb{R}^d$

$$\|x\|_\mathfrak{s} := \sum_{i=1}^{d} |x_i|^{1/\mathfrak{s}_i} . \tag{2.12}$$

and define the scaled unit ball

$$B_\mathfrak{s}(0,1) = \{x \in \mathbb{R}^d \mid \|x\|_\mathfrak{s} < 1\}$$

We also write $|\mathfrak{s}| = \sum_{i=1}^d \mathfrak{s}_i$ and for multi-indices $k \in \mathbb{N}^d$, $|k|_\mathfrak{s} = \sum_{i=1}^d k_i$. In this spirit we also set $|\mathbb{N}^d|_\mathfrak{s} := \{|k|_\mathfrak{s} : \mid k \in \mathbb{N}^d\} \subseteq [0, \infty)$. Finally for a positive $a > 0$ we write

$$a^\mathfrak{s} := \operatorname{diag}(a^{\mathfrak{s}_1}, a^{\mathfrak{s}_2}, \ldots, a^{\mathfrak{s}_d}) , \tag{2.13}$$

where $\operatorname{diag}(\,\cdot\,)$ denotes the diagonal matrix with diagonal " $\cdot$ ". There is also a notion of an anisotropic distance between two sets $A, B \subseteq \mathbb{R}^d$:

$$\operatorname{dist}_\mathfrak{s}(A, B) = \inf\{\|y - x\|_\mathfrak{s} \mid y \in A, \, x \in B\} .$$

Provided $A = \{z\}$ for some $z \in \mathbb{R}^d$ we may write z instead of A in the arguments of $\operatorname{dist}_\mathfrak{s}$ and similar for B. For $A \subseteq \mathbb{R}^d$ we define its anisotropic diameter by

$$\operatorname{diam}_\mathfrak{s}(A) = \sup\{\|x - y\|_\mathfrak{s} \mid x, y \in A\}$$

The *isotropic* formulation of the definitions presented here, which might be more familiar to the reader, arises if we take $\mathfrak{s} = (1, \ldots, 1)$.

The interplay of the scaling vector $\mathfrak{s}$ with the weighted spaces $L^p(\mathbb{R}^d, \rho)$ is described by the following lemma.

Lemma 2.1.10. *Let $F : \mathbb{R}^d \to \mathbb{C}$ be measurable, let $\omega \in \boldsymbol{\omega}$ and $p \in [1, \infty]$. Then, for $\delta \in (0, 1], \lambda > 0$ and any $\lambda' > \lambda$ such that $e^{(\lambda-\lambda')\omega} \in L^p(\mathbb{R}^d)$, we have*

$$\|F^\delta\|_{L^p(\mathbb{R}^d, e^{\lambda\omega(x)})} \lesssim \delta^{-|\mathfrak{s}|(1-1/p)} \cdot p^\omega_{0,\lambda'}(F)\,,$$

where $F^\delta := \delta^{-|\mathfrak{s}|} F(\delta^{-\mathfrak{s}} \cdot)$ (with the matrix *$\delta^{-\mathfrak{s}} = (\delta^{-1})^{\mathfrak{s}}$ defined as in* (2.13)*) and where $p^\omega_{0,\lambda'}$ is as in* (2.5)*. In particular we have for $r, q \in [1, \infty]$ with $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $f \in L^q(\mathbb{R}^d, \rho)$, $\rho \in \boldsymbol{\rho}(\omega)$*

$$\|F^\delta * f\|_{L^r(\mathbb{R}^d, \rho)} \lesssim \|F^\delta\|_{L^p(\mathbb{R}^d, e^{\overline{\lambda}\omega(x)})} \cdot \|f\|_{L^q(\mathbb{R}^d, \rho)} \lesssim \delta^{-|\mathfrak{s}|(1-1/p)} \cdot p^\omega_{0,\overline{\lambda}'}(F)\, \|f\|_{L^q(\mathbb{R}^d, \rho)} \tag{2.14}$$

for some $\overline{\lambda} = \overline{\lambda}(\rho) > 0$ and any $\overline{\lambda}' > \overline{\lambda}$ such that $e^{(\overline{\lambda}-\overline{\lambda}')\omega} \in L^p(\mathbb{R}^d)$.

Proof. For $p < \infty$ we rewrite

$$\begin{aligned}\int |e^{\lambda\,\omega(x)} \delta^{-|\mathfrak{s}|} F(\delta^{-\mathfrak{s}} x)|^p \mathrm{d}x &= \delta^{-|\mathfrak{s}|(p-1)} \int e^{\lambda p\,\omega(\delta^{\mathfrak{s}} x)} |F(x)|^p \mathrm{d}x \\ &\leqslant \delta^{-|\mathfrak{s}|(p-1)} \int e^{\lambda p\,\omega(x)} |F(x)|^p\,,\end{aligned}$$

where we used monotonicity of ω in the last step to drop $\delta^{\mathfrak{s}}$. Using now simply the definition of the seminorm $p^\omega_{0,\lambda'}$ we obtain the first estimate for $p < \infty$. The case $p = \infty$ however is obvious. The convolution inequality is then just a consequence of Young's inequality after an application of (2.2):

$$\left|\rho(y) \int F^\delta(y - x) f(x) \mathrm{d}x\right| \lesssim \int |e^{\overline{\lambda}\,\omega(y-x)} F^\delta(y - x)|\, |\rho(x) f(x)| \mathrm{d}x\,.$$

□

The following types of sets will be the building blocks for our definitions below.

Definition 2.1.11. *We say that a set $\mathcal{B} \subseteq \mathbb{R}^d$ is a box if there are $a_1, \ldots, a_d > 0$ such that $\mathcal{B} = \times_{i=1}^d [-a_i, a_i]$. A set $\mathcal{A} \subseteq \mathbb{R}^d$ is a rectangular annulus if there are two boxes $\mathcal{B}$, $\tilde{\mathcal{B}} \subseteq \mathbb{R}^d$ with $\mathcal{B} \subseteq \tilde{\mathcal{B}}$ and $\partial\mathcal{B} \cap \partial\tilde{\mathcal{B}} = \varnothing$ such that $\mathcal{A} = \overline{\tilde{\mathcal{B}} \backslash \mathcal{B}}$.*

We then have the following elementary properties.

Lemma 2.1.12. *Let $\mathcal{A}, \tilde{\mathcal{A}}$ be two rectangular annuli and let $\mathcal{B}$ be a box. If we define for $j \geqslant 0$ $\mathcal{A}_j := 2^{j\mathfrak{s}}\mathcal{A}$, $\tilde{\mathcal{A}}_j := 2^{j\mathfrak{s}}\tilde{\mathcal{A}}$ and $\mathcal{B}_j := 2^{j\mathfrak{s}}\mathcal{B}$ (with a* matrix *$2^{j\mathfrak{s}}$ as in* (2.13)*) we have the following relations:*

- *If $\mathcal{A}_i \cap \mathcal{B}_j \neq \varnothing$ only if $i \lesssim j$,*
- *If $\mathcal{A}_i \cap \tilde{\mathcal{A}}_j \neq \varnothing$ only if $i \sim j$,*

where $\lesssim$ and $\sim$ should be read as on page 17.

Proof. For the first statement we can write $2^{i\mathfrak{s}}\mathcal{A} \cap 2^{j\mathfrak{s}}\mathcal{B} \neq \varnothing$ as $\mathcal{A} \cap 2^{(j-i)\mathfrak{s}}\mathcal{B} \neq \varnothing$ and use that for $i \gtrsim j$ this cannot be true. The second statement then follows if we use that $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are both contained in a box. □

Let us now fix some choice of *anisotropic* dyadic partition of unity for $\omega \in \boldsymbol{\omega}$. For this purpose we follow [Tri06, Section 5.1] [2]:

Let in the following $\mathcal{B}_{-2} := \varnothing$ and set $\mathcal{B}_j := 2^{(j+1)\mathfrak{s}}[-1,1]^d$ for $j \geqslant -1$. Fix further (via Lemma 2.1.9) a symmetric and positive $\varphi_{-1} \in C^\infty_{\omega,c}(\mathbb{R}^d)$ with values in $[0,1]$ such that $\varphi_{-1} = 1$ on $3/2 \cdot \mathcal{B}_{-1} = [-3/2, 3/2]^d$ and $\operatorname{supp} \varphi_{-1} \subseteq \mathcal{B}_0 = 2^{\mathfrak{s}}[-1,1]^d$. We then set for $j \geqslant 0$

$$\varphi_j = \varphi_{-1}(2^{-(j+1)\mathfrak{s}} \cdot) - \varphi_{-1}(2^{-j\mathfrak{s}} \cdot)\,,$$

(with *matrices* $2^{-j\mathfrak{s}}$, $2^{-(j+1)\mathfrak{s}}$ as in (2.13)) which yields a family $(\varphi_j)_{j \geqslant -1} \in C^\infty_{\omega,c}(\mathbb{R}^d)$ that satisfies the following properties:

- For $j \geqslant 0$
$$\varphi_j = \varphi_0(2^{-j\mathfrak{s}} \cdot)\,. \tag{2.15}$$
- For $j \geqslant 0$
$$\sum_{i<j} \varphi_i = \varphi_{-1}(2^{-j\mathfrak{s}} \cdot)\,. \tag{2.16}$$
- $\varphi_j \geqslant 0$ for $j \geqslant -1$.
- $\sum_{j \geqslant -1} \varphi_j(x) = 1$ for $x \in \mathbb{R}^d$.
- $\operatorname{supp} \varphi_j \subseteq \mathcal{B}_{j+1} \backslash \mathcal{B}_{j-1}$ for $j \geqslant -1$, in particular $\operatorname{supp} \varphi_j \cap \operatorname{supp} \varphi_{j'} = \varnothing$ for $|j - j'| > 1$.

[2] To be precise, [Tri06] works with $\mathfrak{s} \in (0,\infty)^d$, $\sum_{i=1}^d \mathfrak{s}_i = d$ instead, which allows for an interpretation of the Besov regularity in Definition 2.1.16 below as some sort of *mean* regularity. We here follow the scaling that corresponds to the definitions in [Hai14].

Such a family is called an *(anisotropic) dyadic partition of unity.* Note that for $j \geqslant 0$ the support of φ_j is contained in an rectangular annulus of size $2^j\mathcal{A}$, as defined in Definition 2.1.11. The support of φ_{-1} is contained in the box $\mathcal{B}_0$.

Remark 2.1.13. *The choice of the sequence $\mathcal{B}_j$ was of course rather arbitrary. One could for example have chosen another sequence of boxes such as $\tilde{\mathcal{B}}_j = a \cdot \mathcal{B}_j$ with some $a > 0$ instead, which would then lead to a different partition of unity $(\tilde{\varphi}_j)_{j\geqslant -1}$.*

A special role in this paper will be played by the following functions

$$\Psi^j := \mathcal{F}^{-1}_{\mathbb{R}^d}\varphi_j\,, \qquad \Psi^{<j} = \sum_{-1\leqslant i<j} \Psi^i = \mathcal{F}^{-1}_{\mathbb{R}^d}\left(\varphi_{-1}(2^{-j\mathfrak{s}}\cdot)\right) \tag{2.17}$$

with $j \geqslant -1$ (note that $\Psi^{<-1} = 0$). We also use occasionally the notation $\Psi^{\leqslant j} := \Psi^{<j+1}$ for $j \geqslant -1$.

The properties of Ψ^j and $\Psi^{<j}$ are summarized by the following Lemma, which is rather elementary to check and therefore not proved here.

Lemma 2.1.14. *The functions $\Psi^j, \Psi^{<j}$ defined above satisfy*

- *Ψ^j and $\Psi^{<j}$ are symmetric for $j \geqslant -1$.*
- *$\int \mathrm{d}v\, \Psi^j_v \cdot v^k = 0$ for $j \geqslant 0$ and $k \in \mathbb{N}^d$.*
- *$\int \mathrm{d}u\, \Psi^{<j}_u = 1$ for $j \geqslant 0$.*
- *$\Psi^j = 2^{j|\mathfrak{s}|}\phi_1(2^{j\mathfrak{s}}\cdot)$, $\Psi^{<j} = 2^{j|\mathfrak{s}|}\phi_2(2^{j\mathfrak{s}}\cdot)$ for $j \geqslant 0$ and some $\phi_1, \phi_2 \in \mathcal{S}_\omega(\mathbb{R}^d)$.*
- *If for some $a \geqslant 0$ and some measurable $f : \mathbb{R}^d \to \mathbb{C}$ we have $|f_x| \lesssim \|x\|^a_{\mathfrak{s}}$, then this implies*

$$\left|\int \mathrm{d}u\, \Psi^j_u \cdot f_u\right|, \left|\int \mathrm{d}v\, \Psi^{<j} \cdot f_v\right| \lesssim 2^{-ja}$$

for $j \geqslant -1$.

- *Given a $f \in L^\infty(\mathbb{R}^{2d})$ and $j > 0$ the map*

$$x \mapsto \iint \mathrm{d}u\,\mathrm{d}v\, \Psi^{<j-1}_{x-u}\Psi^j_{x-v} f(u,v)$$

is spectrally supported in an rectangular annulus $2^{j\mathfrak{s}}\mathcal{A}$ (with $\mathcal{A}$ independent of f and j).

The action of $\Psi^{<j}$ on polynomials can be stated as follows.

Lemma 2.1.15. *For k, $l \in \mathbb{N}^d$ and $j > -1$ we have*

$$\int \mathrm{d}v \, \partial^k \Psi^{<j}_{-v} \, v^l = \delta_{kl} \, k!$$

Proof. By integration by parts the left hand side equals $\binom{l}{k} k! \int \mathrm{d}v \, \Psi^{<j}_{-v} \, v^{l-k}$ (or 0 if $l > k$). This expression can then be expressed via the inverse Fourier transform as

$$\mathbf{1}_{l \leqslant k} \binom{l}{k} k! \, (2\pi \imath)^{k-l} \partial^{l-k} \mathcal{F}^{-1}_{\mathbb{R}^d} \Psi^{<j}_{-\cdot}(0) \, ,$$

which yields the claim since $\mathcal{F}^{-1}_{\mathbb{R}^d} \Psi^{<j}$ equals 1 in a box around 0. □

Using the functions φ_j or their Fourier transforms Ψ^j we can then define *Littlewood-Paley blocks* for $f \in \mathcal{S}'_\omega(\mathbb{R}^d)$ and $j \geqslant -1$ by

$$\Delta_j f = \mathcal{F}^{-1}_{\mathbb{R}^d}(\varphi_j \cdot \mathcal{F}_{\mathbb{R}^d} f) = \Psi^j * f = \int \mathrm{d}u \, \Psi^j_{\cdot - u} \, f_u \, . \tag{2.18}$$

which are by construction smooth functions. We can decompose any $f \in \mathcal{S}'_\omega(\mathbb{R}^d)$ by its *Littlewood-Paley decomposition*:

$$f = \sum_{j \geqslant -1} \Delta_j f \, , \tag{2.19}$$

where the sum on the right hand side converges in the topology of $\mathcal{S}'_\omega(\mathbb{R}^d)$. Using the scaling of $\Psi^j = 2^{j|\mathfrak{s}|} \phi_1(2^{j\mathfrak{s}} \cdot)$, for $j \geqslant 0$, from Lemma 2.1.14 and the weighted Young inequality in Lemma 2.1.10 we see that the blocks Δ_j map for $j \geqslant -1$ the space $L^p(\mathbb{R}^d, \rho)$ into itself for $p \in [1, \infty]$ and $\rho \in \boldsymbol{\rho}(\omega)$:

$$\|\Delta_j f\|_{L^p(\mathbb{R}^d,\rho)} = \|\Psi^j * f\|_{L^p(\mathbb{R}^d,\rho)} \lesssim \|f\|_{L^p(\mathbb{R}^d,\rho)} \, , \tag{2.20}$$

where the involved constant can be chosen independent of j. We will also use sometimes the following operator

$$S_j f := \sum_{-1 \leqslant i < j} \Delta_i f = \Psi^{<j} * f = \int \mathrm{d}v \, \Psi^{<j}_{\cdot - v} \, f_v \, ,$$

which maps again $L^p(\mathbb{R}^d, \rho)$ into itself by the same argument. Using the decomposition (2.19) we can now define anisotropic and weighted Besov spaces.

Definition 2.1.16. *Let $\omega \in \boldsymbol{\omega}$, $\rho \in \boldsymbol{\rho}(\omega)$, $\gamma \in \mathbb{R}$ and $p, q \in [1, \infty]$. We then define the anisotropic, weighted Besov spaces by*

$$\mathcal{B}^\gamma_{p,q,\mathfrak{s}}(\mathbb{R}^d, \rho) := \left\{ f \in \mathcal{S}'_\omega(\mathbb{R}^d) \,\middle|\, \|f\|_{\mathcal{B}^\gamma_{p,q,\mathfrak{s}}(\mathbb{R}^d,\rho)} < \infty \right\} \, ,$$

where

$$\|f\|_{\mathcal{B}^{\gamma}_{p,q,\mathfrak{s}}(\mathbb{R}^d,\rho)} := \left\| \left(2^{j\gamma} \|\Delta_j f\|_{L^p(\mathbb{R}^d,\rho)}\right)_{j\geqslant -1} \right\|_{\ell^q} .$$

with the Littlewood-Paley blocks $(\Delta_j)_{j\geqslant -1}$ *constructed as above (for the considered* ω*). We also write*

$$\begin{aligned} \mathcal{C}^{\gamma}_{p,\mathfrak{s}}(\mathbb{R}^d,\rho) &:= \mathcal{B}^{\gamma}_{p,\infty,\mathfrak{s}}(\mathbb{R}^d,\rho) \\ \mathcal{C}^{\gamma}_{\mathfrak{s}}(\mathbb{R}^d,\rho) &:= \mathcal{C}^{\gamma}_{\infty,\mathfrak{s}}(\mathbb{R}^d,\rho) = \mathcal{B}^{\gamma}_{\infty,\infty,\mathfrak{s}}(\mathbb{R}^d,\rho) \\ H^{\gamma}_{\mathfrak{s}}(\mathbb{R}^d,\rho) &:= \mathcal{B}^{\gamma}_{2,2}(\mathbb{R}^d,\rho) \end{aligned}$$

The spaces $\mathcal{C}^{\gamma}_{\mathfrak{s}}(\mathbb{R}^d,\rho)$ *are called Hölder-Zygmund spaces, the spaces* $H^{\gamma}_{\mathfrak{s}}(\mathbb{R}^d,\rho)$ *fractional Sobolev spaces.*

In the isotropic case*, that is* $\mathfrak{s} = (1,\dots,1)$*, we will skip the index* $\mathfrak{s}$*, for example* $\mathcal{B}^{\gamma}_{p,q}(\mathbb{R}^d,\rho) = \mathcal{B}^{\gamma}_{p,q,(1,\dots,1)}(\mathbb{R}^d,\rho)$ *and similar for the other spaces above. In the unweighted case, which is* $\rho = 1$*,* ρ *will be skipped in the notation for the corresponding spaces, for example* $\mathcal{B}^{\gamma}_{p,q,\mathfrak{s}}(\mathbb{R}^d) = \mathcal{B}^{\gamma}_{p,q,\mathfrak{s}}(\mathbb{R}^d,1)$ *and similar for the other spaces above.*

Remark 2.1.17. *The reader might be a bit worried that whenever there are* ω, $\omega' \in \boldsymbol{\omega}$ *with* $\boldsymbol{\rho}(\omega) \subseteq \boldsymbol{\rho}(\omega')$ *and* $\rho \in \boldsymbol{\rho}(\omega)$ *the definition of* $\mathcal{B}^{\gamma}_{p,q,\mathfrak{s}}(\mathbb{R}^d,\rho)$ *might depend on whether we chose* ω *or* ω' *in the construction as one has* $\mathcal{S}'_{\omega}(\mathbb{R}^d) \subseteq \mathcal{S}'_{\omega'}(\mathbb{R}^d)$*. However, it turns out this is not the case since if* $f \in \mathcal{S}'_{\omega'}(\mathbb{R}^d)$ *with* $\|f\|_{\mathcal{B}^{\gamma}_{p,q,\mathfrak{s}}(\mathbb{R}^d,\rho)} < \infty$*, with the space* $\mathcal{B}^{\gamma}_{p,q,\mathfrak{s}}$ *constructed from* ω'*, one has for every* $j \geqslant -1$ *that* $\|\Delta_j f\|_{L^p(\mathbb{R}^d,\rho)} < \infty$*. This implies that* $\Delta_j f$ *is an element of* $\mathcal{S}'_{\omega}(\mathbb{R}^d)$ *and so is* f *as the limit of* $\sum_{j=-1}^{N} \Delta_j f$*.*

We see therefore that $\mathcal{B}^{\gamma}_{p,q,\mathfrak{s}}(\mathbb{R}^d,\rho)$ *is identical for* $\omega, \omega' \in \boldsymbol{\omega}$ *with* $\rho \in \boldsymbol{\rho}(\omega) \cap \boldsymbol{\rho}(\omega')$*.*

Remark 2.1.18. *Using Lemma 2.1.12 one sees that another choice of dyadic partition of unity* $(\tilde{\varphi}_j)_{j\geqslant -1} \subseteq C^{\infty}_{\omega,c}(\mathbb{R}^d)$ *instead of* $(\varphi_j)_{j\geqslant -1}$ *gives an equivalent norm for* $\mathcal{B}^{\gamma}_{p,q}(\mathbb{R}^d,\rho)$*.*

Occasionally we will write

$$\mathcal{B}^{\infty}_{p,q,\mathfrak{s}}(\mathbb{R}^d,\rho) = \bigcap_{\gamma\in\mathbb{R}} \mathcal{B}^{\gamma}_{p,q,\mathfrak{s}}(\mathbb{R}^d,\rho) .$$

We will use a similar notation for the other spaces above.

We have the following straightforward modification of [GIP15, Lemma A.3].

Lemma 2.1.19. *Given a sequence of smooth* $(f_j)_{j\geqslant -1}$ *such that* $\operatorname{supp} \mathcal{F}_{\mathbb{R}^d} f_j \subseteq 2^{j\mathfrak{s}}\mathcal{B}$ *for some box* $\mathcal{B}$*, we have for* $\gamma > 0$ *and* $f := \sum_{j\geqslant -1} f_j$

$$\|f\|_{\mathcal{B}^{\gamma}_{p,q}(\mathbb{R}^d)} \lesssim \left\| \left(2^{j\gamma} \|f_j\|_{L^p(\mathbb{R}^d)}\right)_{j\geqslant -1} \right\|_{\ell^q} . \tag{2.21}$$

If $\operatorname{supp} \mathcal{F}_{\mathbb{R}^d} f_j \subseteq 2^{j\mathfrak{s}}\mathcal{A}$ *for some rectangular annulus* $\mathcal{A}$*, then* (2.21) *is true for general* $\gamma \in \mathbb{R}$*.*

An intuition behind the anisotropic scaling is that an object $f \in \mathcal{B}^{\gamma}_{p,q,\mathfrak{s}}(\mathbb{R}^d)$ has "smoothness $\alpha/\mathfrak{s}_i$ in direction $i \in \{1,\dots,d\}$" [3] To strenghten this intuition we will find a different characterization of the Hölder-Zygmund spaces $\mathcal{C}^{\gamma}_{\mathfrak{s}}$ based on the Taylor remainder for $\gamma > 0$

$$R^{\gamma}_{x;h} f := f - T^{\gamma}_{x;h} f\,, \tag{2.22}$$

where

$$T^{\gamma}_{x;h} f := \sum_{k \in \mathbb{N}^d_{<\gamma}} \frac{1}{k!} \partial^k f(x)\, h^k$$

for $x, h \in \mathbb{R}^d$, $\gamma > 0$, $\mathbb{N}^d_{<\gamma} := \{k \in \mathbb{N}^d \,|\, |k|_{\mathfrak{s}} < \gamma\}$ and with f having enough derivatives such that these expressions make sense. $R^{\gamma}_{x;h} f$ can be rewritten by an application of Proposition A.1 of [Hai14].

Lemma 2.1.20. *Let $\Omega \subseteq \mathbb{R}^d$ be open and convex, let $\gamma \in (0,\infty)\backslash|\mathbb{N}^d|_{\mathfrak{s}}$ and let $f : \Omega \to \mathbb{C}$ be smooth. We then have for $x, h \in \mathbb{R}^d$*

$$R^{\gamma}_{x;h} f = \sum_{k \in \mathbb{N}^d_{>\gamma}} R^{\gamma,k}_{x;h} f = \sum_{k \in \mathbb{N}^d_{>\gamma}} \frac{h^k}{(k - e_{\mathfrak{m}(k)})!} \int_0^1 \mathrm{d}t\, \partial^k f(x + v^k_t(h))\, (1-t)^{k_{\mathfrak{m}(k)}-1}\,, \tag{2.23}$$

where $\mathfrak{m}(k) = \min\{j \,|\, k_j \neq 0\}$, $\mathbb{N}^d_{>\gamma} := \{k \in \mathbb{N}^d \,|\, |k|_{\mathfrak{s}} > \gamma,\ |k - e_{\mathfrak{m}(k)}|_{\mathfrak{s}} < \gamma\}$ *and*

$$v^k_t(h) = (h_1, \dots, h_{\mathfrak{m}(k)-1}, t \cdot h_{\mathfrak{m}(k)}, 0, \dots, 0)\,.$$

Remark 2.1.21. *The set $\mathbb{N}^d_{>\gamma}$ can be thought of as the "discrete boundary" of $\mathbb{N}^d_{<\gamma}$. Note that this set is really* finite.

Remark 2.1.22. *We also extend the notation $R^{\gamma}_{x;h} f$ to $\gamma < 0$ by simply setting in this case $R^{\gamma}_{x;h} f = f(x+h)$ for $f : \Omega \to \mathbb{C}$.*

Then the announced characterization of the anisotropic Hölder-Zygmund spaces is described by the following Lemma, which is a modification of [BCD11, Theorem 2.36].

[3] Although this intuition is helpful to "guess" $\mathfrak{s}$ in many situations it is actually slightly incorrect, since the parameter α should really be read in the sense of an average . A more appropriate (but rather useless) intuition for $\mathcal{B}^{\alpha}_{p,q,\mathfrak{s}}$ would be that $f \in \mathcal{B}^{\alpha}_{p,q,\mathfrak{s}}$ has in average a smoothness of $d \cdot \alpha/|\mathfrak{s}|$. Compare the regularity of white noise in Lemma 2.2.2 below as an example where the "directional intuition" evidently fails.

Lemma 2.1.23. *For $\gamma \in (0,\infty)\backslash|\mathbb{N}^d|_{\mathfrak{s}}$ and $\rho \in \boldsymbol{\rho}(\boldsymbol{\omega})$ an equivalent norm for $\mathcal{C}^{\gamma}_{\mathfrak{s}}(\mathbb{R}^d,\,\rho)$ is given by*

$$\sup_{l\in\mathbb{N}^d_{<\gamma}}\sup_{x\in\mathbb{R}^d}\rho(x)|\partial^l f(x)| + \sup_{l\in\mathbb{N}^d_{<\gamma}}\sup_{x,y\in\mathbb{R}^d,\,0<\|x-y\|_{\mathfrak{s}}\leqslant 1}\rho(x)\frac{|\partial^l f(y) - T^{\gamma-|l|_{\mathfrak{s}}}_{x;y-x}\partial^l f|}{\|y-x\|_{\mathfrak{s}}^{\gamma-|l|_{\mathfrak{s}}}}\,. \tag{2.24}$$

Proof. Assume that $f \in \mathcal{C}^{\gamma}_{\mathfrak{s}}(\mathbb{R}^d,\rho)$ as defined in Definition 2.1.16 above and further, without loss of generality, that $\|f\|_{\mathcal{C}^{\gamma}_{\mathfrak{s}}(\mathbb{R}^d,\rho)} \leqslant 1$. If we write $\overline{\Delta}_j f := \sum_{i:\,|i-j|\leqslant 1}\Delta_i f$, we have by spectral support properties $\Delta_j f = \Delta_j\overline{\Delta}_j f = \Psi^j * \overline{\Delta}_j f$. Indeed, by Definition of the blocks Δ_j and by our choice of supports of φ we have

$$\mathcal{F}_{\mathbb{R}^d}(\Delta_j\overline{\Delta}_j f) = \varphi_j\sum_{i:\,|j-i|\leqslant 1}\varphi_i\,\mathcal{F}_{\mathbb{R}^d}f = \varphi_j\sum_{i\geqslant -1}\varphi_i\,\mathcal{F}_{\mathbb{R}^d}f = \varphi_j\,\mathcal{F}_{\mathbb{R}^d}f = \mathcal{F}_{\mathbb{R}^d}(\Delta_j f)\,,$$

from which the claimed identity follows. Thus for $l \in \mathbb{N}^d_{<\gamma}$ by the weighted Young inequality (2.14) in Lemma 2.1.10

$$\begin{aligned}\|\partial^l\Delta_j f\|_{L^\infty(\mathbb{R}^d,\rho)} &= \|\partial^l\Psi^j * \overline{\Delta}_j f\|_{L^\infty(\mathbb{R}^d,\rho)} \lesssim \|\partial^l\Psi^j\|_{L^1(\mathbb{R}^d,e^{\overline{\lambda}\omega})}\,\|\overline{\Delta}_j f\|_{L^\infty(\mathbb{R}^d,\rho)}\\ &\lesssim 2^{j|l|_{\mathfrak{s}}}2^{-j\gamma} = 2^{-j(\gamma-|l|_{\mathfrak{s}})}\,,\end{aligned} \tag{2.25}$$

where $\overline{\lambda}$ is as in Lemma 2.1.10 and where we applied once more Lemma 2.1.10 and Lemma 2.1.14 to estimate $\|\partial^l\Psi^j\|_{L^1(\mathbb{R}^d,e^{\overline{\lambda}\omega})} \lesssim 2^{j|l|_{\mathfrak{s}}}$. This implies that the first term of (2.24) is bounded by 1 via

$$\sup_{x\in\mathbb{R}^d}\rho(x)|\partial^l f(x)| \leqslant \sup_{x\in\mathbb{R}^d}\sum_{j\geqslant -1}\rho(x)|\partial^l\Delta_j f(x)| \lesssim \sum_{j>-1}2^{-j(\gamma-|l|_{\mathfrak{s}})} \lesssim 1\,.$$

To bound the second term of (2.24) we first consider for $j \geqslant -1$ and $x,\,y \in \mathbb{R}^d$ with $0 < \|x-y\|_{\mathfrak{s}} \leqslant 1$ the expansion of the Littlewood-Paley blocks

$$\begin{aligned}&(\partial^l\Delta_j f)_y - \sum_{k\in\mathbb{N}^d_{<\gamma-|l|_{\mathfrak{s}}}}\frac{(\partial^{k+l}\Delta_j f)_x}{k!}(y-x)^k\\ &= \int \mathrm{d}u\left(\partial^l\Psi^j_{y-u} - \sum_{k\in\mathbb{N}^d_{<\gamma-|l|_{\mathfrak{s}}}}\frac{\partial^{k+l}\Psi^j_{x-u}}{k!}(y-x)^k\right)(\overline{\Delta}_j f)_u\,,\end{aligned}$$

where we used once more that $\Delta_j\overline{\Delta}_j f = \Delta_j f$ for $\overline{\Delta}_j f = \sum_{i:\,|i-j|\leqslant 1}\Delta_i f$ as above. Applying (2.23) then gives

$$\sum_{k\in\mathbb{N}^d_{>\gamma-|l|_{\mathfrak{s}}}}\frac{(x-y)^k}{(k-e_{\mathfrak{m}(k)})!}\int_0^1\mathrm{d}t\int\mathrm{d}u\,\partial^{k+l}\Psi^j_{x-u+v^k_t(y-x)}(1-t)^{k_{\mathfrak{m}(k)}-1}\,(\overline{\Delta}_j f)_u\,.$$

Using that $\rho_x/\rho_{x+v_t^k(x-y)} \lesssim 1$ by (2.2) and $\|v_t^k(y-x)\|_{\mathfrak{s}} \leqslant \|y-x\|_{\mathfrak{s}} \leqslant 1$, we obtain together with the weighted Young inequality (2.14) (in Lemma 2.1.10) the bound

$$\begin{aligned}
&\rho_x\Big|(\partial^l \Delta_j f)_y - \sum_{k\in\mathbb{N}^d_{<\gamma-|l|_{\mathfrak{s}}}} \frac{(\partial^{k+l}\Delta_j f)_x}{k!}(y-x)^k\Big| \\
&\lesssim \sum_{k\in\mathbb{N}^d_{>\gamma-|l|_{\mathfrak{s}}}} \|y-x\|_{\mathfrak{s}}^{|k|_{\mathfrak{s}}} \int_0^1 \mathrm{d}t\, \rho_{x+v_t^k(y-x)} \,|(\partial^{k+l}\Psi^j * \overline{\Delta}_j f)_{x+v_t^k(y-x)}| \qquad (2.26)\\
&\lesssim \sum_{k\in\mathbb{N}^d_{>\gamma-|l|_{\mathfrak{s}}}} \|y-x\|_{\mathfrak{s}}^{|k|_{\mathfrak{s}}} \int_0^1 \mathrm{d}t\, \|\overline{\Delta}_j f\|_{L^\infty(\mathbb{R}^d,\rho)} \,\|\partial^{k+l}\Psi^j_{x-\cdot+v_t^k(y-x)}\|_{L^1(\mathbb{R}^d,e^{\overline{\lambda}\omega})}\\
&\lesssim \sum_{k\in\mathbb{N}^d_{>\gamma-|l|_{\mathfrak{s}}}} \|x-y\|_{\mathfrak{s}}^{|k|_{\mathfrak{s}}} 2^{j(|k|_{\mathfrak{s}}+|l|_{\mathfrak{s}}-\gamma)} \qquad (2.27)
\end{aligned}$$

where $\overline{\lambda}$ is as in (2.14) and where we applied $\|\overline{\Delta}_j f\|_{L^\infty(\mathbb{R}^d,\rho)} \lesssim 2^{-j\gamma}$ and

$$\|\partial^{k+l}\Psi^j_{x-\cdot+v_t^k(y-x)}\|_{L^1(\mathbb{R}^d,e^{\overline{\lambda}\omega})} \lesssim 2^{j(|k|_s+|l|_{\mathfrak{s}})}$$

in the second step.

On the other hand using $\rho_x/\rho_y \lesssim 1$ we can directly estimate via (2.25)

$$\begin{aligned}
&\rho_x\Big|(\partial^l \Delta_j f)_y - \sum_{k\in\mathbb{N}^d_{<\gamma-|l|_{\mathfrak{s}}}} \frac{(\partial^{k+l}\Delta_j f)_x}{k!}(y-x)^k\Big| \\
&\lesssim \left(2^{-j(\gamma-|l|_{\mathfrak{s}})} + \sum_{k\in\mathbb{N}^d_{<\gamma-|l|_{\mathfrak{s}}}} 2^{-j(\gamma-|l|_{\mathfrak{s}}-|k|_{\mathfrak{s}})}\|y-x\|_{\mathfrak{s}}^{|k|_{\mathfrak{s}}}\right). \qquad (2.28)
\end{aligned}$$

Next, we decompose the Taylor expansion in a "low-frequency" and a "high-frequency" term. That is, choose $j' \geqslant -1$ such that $2^{-j'-1} < \|y-x\|_{\mathfrak{s}} \leqslant 2^{-j'}$ and split

$$\begin{aligned}
\partial^l f_y - \sum_{k\in\mathbb{N}^d_{<\gamma-|l|_{\mathfrak{s}}}} \frac{(\partial_{k+l} f)_x}{k!}(y-x)^k &= \sum_{j:\, j\leqslant j'} \left[(\partial^l \Delta_j f)_y - \sum_{k\in\mathbb{N}^d_{<\gamma-|l|_{\mathfrak{s}}}} \frac{(\partial^{k+l}\Delta_j f)_x}{k!}(y-x)^k\right]\\
&\quad+ \sum_{j:\, j>j'} \left[(\partial^l \Delta_j f)_y - \sum_{k\in\mathbb{N}^d_{<\gamma-|l|_{\mathfrak{s}}}} \frac{(\partial^{k+l}\Delta_j f)_x}{k!}(y-x)^k\right].
\end{aligned}$$

Applying now (2.27) to the first and (2.28) to the second term yields the first direction on the equivalence:

$$
\begin{aligned}
&\rho_x \left| \partial^l f_y - \sum_{k \in \mathbb{N}^d_{<\gamma - |l|_\mathfrak{s}}} \frac{(\partial^{k+l} f)_x}{k!} (y-x)^k \right| \\
&\lesssim \sum_{j\colon j \leqslant j'} \sum_{k \in \mathbb{N}^d_{>\gamma - |l|_\mathfrak{s}}} \|x-y\|_\mathfrak{s}^{|k|_\mathfrak{s}} 2^{j(|k|_\mathfrak{s} + |l|_\mathfrak{s} - \gamma)} \\
&+ \sum_{j\colon j > j'} \Big(2^{-j(\gamma - |l|_\mathfrak{s})} + \sum_{k \in \mathbb{N}^d_{<\gamma - |l|_\mathfrak{s}}} 2^{-j(\gamma - |l|_\mathfrak{s} - |k|_\mathfrak{s})} \|y-x\|_\mathfrak{s}^{|k|_\mathfrak{s}} \Big) \\
&\lesssim \sum_{k \in \mathbb{N}^d_{>\gamma - |l|_\mathfrak{s}}} \|x-y\|_\mathfrak{s}^{|k|_\mathfrak{s}} 2^{j'(|k|_\mathfrak{s} + |l|_\mathfrak{s} - \gamma)} + \Big(2^{-j'(\gamma - |l|_\mathfrak{s})} + \sum_{k \in \mathbb{N}^d_{<\gamma - |l|_\mathfrak{s}}} 2^{-j'(\gamma - |l|_\mathfrak{s} - |k|_\mathfrak{s})} \|y-x\|_\mathfrak{s}^{|k|_\mathfrak{s}} \Big) \\
&\lesssim \|y-x\|_\mathfrak{s}^{\gamma - |l|_\mathfrak{s}} \|f\|_{\mathcal{C}^\gamma} \,.
\end{aligned}
$$

For the opposite direction suppose that (2.24) is bounded by 1 without loss of generality. We reshape for $j > -1$ using Lemma 2.1.14

$$
\begin{aligned}
\rho_x |(\varDelta_j f)_x| &= \rho_x \Big| \int \mathrm{d}u\, \Psi^j_{x-u}\, f_u \Big| = \rho_x \Big| \int \mathrm{d}u\, \Psi^j_{x-u} \Big(f_u - \sum_{k \in \mathbb{N}^d_{<\gamma}} \frac{(\partial^k f)_x}{k!} (u-x)^k \Big) \Big| \\
&\leqslant \rho_x \int_{u\colon \|u-x\|_\mathfrak{s} \leqslant 1} \mathrm{d}u\, |\Psi^j_{x-u}| |R^\gamma_{x;u-x} f| + \rho_x \int_{u\colon \|u-x\|_\mathfrak{s} > 1} \mathrm{d}u\, |\Psi^j_{x-u}| |R^\gamma_{x;u-x} f|
\end{aligned}
$$

The first term can be bounded using (2.2) and Lemma 2.1.14:

$$
\begin{aligned}
\int_{u\colon \|u-x\|_\mathfrak{s} \leqslant 1} \mathrm{d}u\, |\Psi^j_{x-u}| \frac{\rho_x}{\rho_u} \cdot \rho_u |R^\gamma_{x;u-x} f| &\lesssim \int_{u\colon \|u-x\|_\mathfrak{s} \leqslant 1} \mathrm{d}u\, |\Psi^j_{x-u}|\, \|u-x\|_\mathfrak{s}^\gamma \\
&\leqslant \int_{\mathbb{R}^d} \mathrm{d}u\, |\Psi^j_{x-u}|\, \|u-x\|_\mathfrak{s}^\gamma \lesssim 2^{-j\gamma} \,.
\end{aligned}
$$

To bound the second term we use once more (2.2) and introduce factors $\|u - x\|^{|k|_s}$, $\|u-x\|_\mathfrak{s}^\gamma \geqslant 1$:

$$
\begin{aligned}
&\int_{u\colon \|u-x\|_\mathfrak{s} > 1} \mathrm{d}u \Big(|\Psi^j_{x-u}| e^{\lambda\, \omega(x-u)}\, |\rho_u f_u| + \sum_{k \in \mathbb{N}^d_{<\gamma}} |\Psi^j_{x-u}| |\rho_x\, (\partial^k f)_x| \cdot \|u-x\|_\mathfrak{s}^{\gamma - |k|_\mathfrak{s}} \Big) \\
&\leqslant \int_{u\colon \|u-x\|_\mathfrak{s} > 1} \mathrm{d}u\, |\Psi^j_{x-u}|\, (e^{\lambda\, \omega(x-u)} + 1)\, \|u-x\|_\mathfrak{s}^\gamma \\
&\lesssim \int_{\mathbb{R}^d} \mathrm{d}u\, |\Psi^j_{x-u}|\, e^{\lambda\, \omega(u-x)}\, \|u-x\|_\mathfrak{s}^\gamma \lesssim 2^{-j\gamma} \,,
\end{aligned}
$$

where we applied in the last step Lemma 2.1.10 for $F = \|\cdot\|_{\mathfrak{s}}^{\gamma}\phi_1$, with ϕ_1 as in Lemma 2.1.14. With (2.20) we get $\|\Delta_{-1}f\|_{L^\infty(\mathbb{R}^d,\rho)} \lesssim \|f\|_{L^\infty(\mathbb{R}^d,\rho)}$ and the desired bound in $\mathcal{C}_{\mathfrak{s}}^{\gamma}(\mathbb{R}^d,\rho)$ follows. □

Lemma 2.1.23 proposes how one can define a local, anisotropic Hölder-Zygmund space.

Definition 2.1.24. *For an open set $\Omega \subset \mathbb{R}^d$ and $\gamma \in (0,\infty)\backslash|\mathbb{N}^d|_{\mathfrak{s}}$ we write $f \in \mathcal{C}^\gamma(\Omega)$ if for some $f : \Omega \to \mathbb{C}$ the norm*

$$\|f\|_{\mathcal{C}^\gamma(\Omega)} := \sup_{l\in\mathbb{N}^d_{<\gamma}} \sup_{x\in\Omega} |\partial^l f(x)| + \sup_{l\in\mathbb{N}^d_{<\gamma}} \sup_{x,y\in\Omega,\, 0<\|y-x\|_{\mathfrak{s}}\leqslant 1} \frac{|\partial^l f_y - T^{\gamma-|l|_{\mathfrak{s}}}_{x;y-x}\partial^l f|}{\|y-x\|_{\mathfrak{s}}^{\gamma-|l|_{\mathfrak{s}}}}$$

is finite

Remark 2.1.25. *Any $f \in \mathcal{C}^\gamma(\Omega)$ and its norm can be immediately extended to boundary points of Ω, in this sense we can also write $\|f\|_{\mathcal{C}^\gamma(\tilde{\Omega})}$ for $\Omega \subseteq \tilde{\Omega} \subseteq \overline{\Omega}$.*

One could als define $\|f\|_{\mathcal{C}^\gamma(\Omega)} := \inf_{g\in\mathcal{C}^\gamma(\mathbb{R}^d):\, g|_\Omega = f} \|g\|_{\mathcal{C}^\gamma(\mathbb{R}^d)}$, which gives an equivalent norm. This is probably a standard result even for anisotropic spaces, but it also follows from our Whitney extension Theorem 5.3.16 we show below. For $\gamma < 0$ one can use this as a definition of $\mathcal{C}^\gamma(\Omega)$.

Similar as in Lemma 2.1.23 one can give an alternative characterization for the anisotropic Sobolev spaces $H_{\mathfrak{s}}^{\gamma}$, compare [Tri06] and Chapter 8 below. We will however only use the isotropic spaces $H^\gamma = H^\gamma_{(1,\dots,1)}$ in this thesis.

We see that the Besov spaces $\mathcal{B}^\gamma_{p,q,\mathfrak{s}}(\mathbb{R}^d,\rho)$ summarize some more "classical" spaces in one framework. They satisfy the well-known *Besov embedding* relations, which in a way generalize the results about Sobolev embedding. Results like this are well-known [Tri06, BCD11], but since we are working with weighted and anisotropic spaces we will give a self-containing proof here.

Lemma 2.1.26. *Let p_1, p_2, q_1, $q_2 \in [1,\infty]$, $\omega \in \boldsymbol{\omega}$, ρ_1, $\rho_2 \in \boldsymbol{\rho}(\omega)$ with $p_1 \leqslant p_2$, $q_1 \leqslant q_2$ and $\rho_2 \lesssim \rho_1$. If $\gamma_1, \gamma_2 \in \mathbb{R}$ are such that $\gamma_2 - \frac{|\mathfrak{s}|}{p_2} \leqslant \gamma_1 - \frac{|\mathfrak{s}|}{p_1}$, we have the continuous embedding*

$$\mathcal{B}^{\gamma_1}_{p_1,q_1,\mathfrak{s}}(\mathbb{R}^d,\rho_1) \subseteq \mathcal{B}^{\gamma_2}_{p_2,q_2,\mathfrak{s}}(\mathbb{R}^d,\rho_2)\,,$$

If we have $\gamma_2 - \frac{d}{p_2} < \gamma_1 - \frac{d}{p_1}$ and $\lim_{|x|\to\infty} \frac{\rho_2(x)}{\rho_1(x)} = 0$ the embedding is compact.

Proof. The continuous embedding follows from the fact that we can write by spectral support properties as in the proof of Lemma 2.1.23

$$\Delta_j f = \sum_{j':\, |j'-j|\leqslant 1} \Delta_{j'}\Delta_j f\,,$$

so that an application of Lemma 2.1.10 to $\sum_{j':|j'-j|\leqslant 1}\Phi^{j'}$ yields

$$\|\Delta_j f\|_{L^{p_2}(\mathbb{R}^d,\rho_2)} \lesssim 2^{j|\mathfrak{s}|(1/p_1-1/p_2)}\|\Delta_j f\|_{L^{p_1}(\mathbb{R}^d,\rho_2)} \lesssim 2^{j|\mathfrak{s}|(1/p_1-1/p_2)}\|\Delta_j f\|_{L^{p_1}(\mathbb{R}^d,\rho_1)}\,, \tag{2.29}$$

from which one easily concludes $\|f\|_{\mathcal{B}^{\gamma_2}_{p_2,q_2}(\mathbb{R}^d(\rho_2))} \lesssim \|f\|_{\mathcal{B}^{\gamma_1}_{p_1,q_1}(\mathbb{R}^d(\rho_1))}$ using $\ell^{q_1}\subseteq\ell^{q_2}$.

To see the compact embedding take a bounded sequence $(f_n)_{n\geqslant 0}\subseteq\mathcal{B}^{\gamma_1}_{p_1,q_1,\mathfrak{s}}(\mathbb{R}^d,\rho_1)$ and fix $\gamma_2'\in(\gamma_2,\gamma_1)$ such that we still have $\gamma_2'-\frac{|\mathfrak{s}|}{p_2}<\gamma_1-\frac{|\mathfrak{s}|}{p_1}$. By Sobolev embedding (or Arzelá-Ascoli if $p_1=\infty$) for the smooth Littlewood-Paley-blocks and the fact that $\lim_{|x|\to\infty}\frac{\rho_2(x)}{\rho_1(x)}=0$ we can find for each $j\geqslant -1$ a convergent subsequence in $L^{p_2}(\mathbb{R}^d,\rho_2)$

$$\varDelta_j f_n \to \delta_j f \tag{2.30}$$

where the limit $\delta_j f$ is in particular spectrally supported in an rectangular annulus $2^{j\mathfrak{s}}\mathcal{A}$. By a diagonalization argument we can pick f_n in (2.30) uniformly in j, and can thus define $f := \sum_{j\geqslant -1}\delta_j f$. By Fatou and the continuous embedding $\mathcal{B}^{\gamma_1}_{p_1,q_1,\mathfrak{s}}(\mathbb{R}^d,\rho_1)\subseteq\mathcal{B}^{\gamma_2'}_{p_2,q_2,\mathfrak{s}}(\mathbb{R}^d,\rho_2)$ we see that indeed $f\in\mathcal{B}^{\gamma_2'}_{p_2,q_2,\mathfrak{s}}(\mathbb{R}^d,\rho_2)$. Using $2^{j\gamma_2}(\|\varDelta_j f_n\|_{L^{p_2}(\mathbb{R}^d,\rho_2)}+\|\varDelta_j f\|_{L^{p_2}(\mathbb{R}^d,\rho_2)})\lesssim 2^{-j(\gamma_2'-\gamma_2)}$ and the relation 2.30 one can then easily make the norm of $f-f_n$ in $\mathcal{B}^{\gamma_2}_{p_2,q_2,\mathfrak{s}}(\mathbb{R}^d,\rho_2)$ smaller than any $\varepsilon>0$ for n large enough. □

Note that Lemma 2.1.26 is only true for $q_1\leqslant q_2$. However, one can also consider $q_1\geqslant q_2$ if one slightly decreases the regularity.

Lemma 2.1.27. *For $p,q_1,q_2\in[1,\infty]$, $\kappa>0$, $\gamma\in\mathbb{R}$ and a weight $\rho\in\boldsymbol{\rho}(\boldsymbol{\omega})$ one has the continuous embedding*

$$\mathcal{B}^{\gamma}_{p,q_1,\mathfrak{s}}(\mathbb{R}^d,\rho)\subseteq\mathcal{B}^{\gamma-\kappa}_{p,q_2,\mathfrak{s}}(\mathbb{R}^d,\rho)$$

Proof. If $q_1\leqslant q_2$ apply Lemma 2.1.26. If $q_1>q_2$ choose $r\in[1,\infty)$ such that $1/q_2=1/q_1+1/r$. The claim then follows by applying Hölder's inequality to $f\in\mathcal{B}^{\gamma}_{p,q_1}(\mathbb{R}^d,\rho)$

$$\|f\|_{\mathcal{B}^{\gamma-\kappa}_{p,q_2,\mathfrak{s}}(\mathbb{R}^d,\rho)} = \left\|\left(2^{-j\kappa}\cdot 2^{j\gamma}\|\Delta_j f\|_{L^p(\mathbb{R}^d,\rho)}\right)_{j\geqslant -1}\right\|_{\ell^{q_2}} \leqslant \|\left(2^{-\kappa}\right)_{j\geqslant -1}\|_{\ell^r}\cdot\|f\|_{\mathcal{B}^{\gamma}_{p,q_1,\mathfrak{s}}(\mathbb{R}^d,\rho)}\,.$$

□

For $\gamma>0$ the Besov space $\mathcal{B}^{\gamma}_{p,q,\mathfrak{s}}$ can further always be embedded in L^p.

Lemma 2.1.28. *For $\gamma>0$, $p,q\in[1,\infty]$ and $\rho\in\boldsymbol{\rho}(\boldsymbol{\omega})$ one has the continuous embedding*

$$\mathcal{B}^{\gamma}_{p,q}(\mathbb{R}^d,\rho)\subseteq L^p(\mathbb{R}^d,\rho)$$

Proof. Take $f \in \mathcal{B}^{\gamma}_{p,q,\mathfrak{s}}(\mathbb{R}^d,\rho)$ and assume without restriction $\|f\|_{\mathcal{B}^{\gamma}_{p,q,\mathfrak{s}}(\mathbb{R}^d,\rho)} \leqslant 1$. Choose $\kappa > 0$ such that $\gamma - \kappa > 0$ and recall from Lemma 2.1.27 that one has

$$\|f\|_{\mathcal{C}^{\gamma-\kappa}_{p,\mathfrak{s}}(\mathbb{R}^d,\rho)} \lesssim \|f\|_{\mathcal{B}^{\gamma}_{p,q,\mathfrak{s}}(\mathbb{R}^d,\rho)} \leqslant 1$$

And thus

$$\|f\|_{L^p(\mathbb{R}^d,\rho)} = \Big\|\sum_{j\geqslant -1} \varDelta_j f\Big\|_{L^p(\mathbb{R}^d,\rho)} \leqslant \sum_{j\geqslant -1} \|\varDelta_j f\|_{L^p(\mathbb{R}^d,\rho)} \lesssim \sum_{j\geqslant -1} 2^{-j(\gamma-\kappa)} \lesssim 1\,.$$

□

Another important property which we need from time to time is the duality of Besov spaces, described in the following Lemma.

Lemma 2.1.29. *[ST87, Section 5.1.2] Let $1 \leqslant p,q < \infty$ and $\gamma \in \mathbb{R}$ and let $\rho \in \boldsymbol{\rho}(\boldsymbol{\omega})$. We then have*

$$\big(\mathcal{B}^{\gamma}_{p,q,\mathfrak{s}}(\mathbb{R}^d;\rho)\big)' = \mathcal{B}^{-\gamma}_{p',q',\mathfrak{s}}(\mathbb{R}^d,\rho^{-1}) \tag{2.31}$$

where $p',q' \in [1,\infty]$ are such that $\frac{1}{p}+\frac{1}{p'} = 1$ and $\frac{1}{q}+\frac{1}{q'} = 1$.

Remark 2.1.30. *[ST87, Section 5.1.2] is only for isotropic spaces, but the modification of the proof (which is actually the one of [Tri83, Theorem 2.11.2]) for the anisotropic case is straightforward.*

In the case $p = \infty$ or $q = \infty$ the equality (2.31) *is wrong, one should in this case replace the right hand side instead by the corresponding completion of $\mathcal{S}_\omega(\mathbb{R}^d)$, compare [Tri83, Remark 2.11.2.2].*

One also has a natural interpolation result for Besov spaces.

Lemma 2.1.31. *Let p_0, p_1, q_0, $q_1 \in [1,\infty]$, $\gamma_0,\gamma_1 \in \mathbb{R}$ and ρ_1, $\rho_2 \in \boldsymbol{\rho}(\omega)$ with $\omega \in \boldsymbol{\omega}$ and assume further that p, q, γ and $\rho \in \boldsymbol{\rho}(\omega)$ are such that there is a $\Theta \in [0,1]$ with $\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$, $\gamma = (1-\Theta)\gamma_0 + \Theta\gamma_1$, $\frac{1}{q} = \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}$ and $\rho = \rho_0^{1-\Theta}\,\rho_1^{\Theta}$. It then holds*

$$\|f\|_{\mathcal{B}^{\gamma}_{p,q,\mathfrak{s}}(\mathbb{R}^d,\rho)} \leqslant \|f\|^{1-\Theta}_{\mathcal{B}^{\gamma_0}_{p_0,q_0,\mathfrak{s}}(\mathbb{R}^d,\rho_1)} \|f\|^{\Theta}_{\mathcal{B}^{\gamma_1}_{p_1,q_1,\mathfrak{s}}(\mathbb{R}^d,\rho_2)}\,. \tag{2.32}$$

Remark 2.1.32. *We should mention at this point that if $p_0 \vee q_0 = p_1 \vee q_1 = \infty$ the space $\mathcal{B}^{\gamma}_{p,q,\mathfrak{s}}(\mathbb{R}^d,\rho)$ is actually* not *the (complex) interpolation space of $\mathcal{B}^{\gamma_0}_{p_0,q_0,\mathfrak{s}}(\mathbb{R}^d,\rho_1)$ and $\mathcal{B}^{\gamma_1}_{p_1,q_1,\mathfrak{s}}(\mathbb{R}^d,\rho_2)$ (compare [SSV14]). However, the estimate* (2.32) *is still true in this case.*

Proof. We follow the ideas from [SSV14, Lemma 3.8]. We assume without loss of generality $0 < \Theta < 1$ and set $r = \frac{p_0}{1-\Theta}$, $r' = \frac{p_1}{\Theta}$ so that

$$\frac{1}{p} = \frac{1}{r} + \frac{1}{r'} .$$

By our definition of $L^p(\mathbb{R}^d, \rho)$ and by application of Hölder's inequality we then obtain for $j \geqslant -1$

$$\begin{aligned}\|\Delta_j f\|_{L^p(\mathbb{R}^d,\rho)} &= \|\Delta_j f\, \rho\|_{L^p(\mathbb{R}^d)} = \|(\Delta_j f)^{1-\Theta}\, \rho_0^{1-\Theta} \cdot (\Delta_j f)^{\Theta}\, \rho_1^{\Theta}\|_{L^p(\mathbb{R}^d)} \\ &\leqslant \|\Delta_j f\, \rho_0\|_{L^{p_0}(\mathbb{R}^d)}^{1-\Theta} \|\Delta_j f\, \rho_1\|_{L^{p_1}(\mathbb{R}^d)}^{\Theta} = \|\Delta_j f\|_{L^{p_0}(\mathbb{R}^d,\rho_0)}^{1-\Theta} \|\Delta_j f\|_{L^{p_1}(\mathbb{R}^d,\rho_1)}^{\Theta} ,\end{aligned}$$

applying then once more Hölder's inequality, but now on ℓ^q, proves the claim. □

Let us finally bookkeep the effect of a derivative on elements in $\mathcal{B}^{\gamma}_{p,q,\mathfrak{s}}(\mathbb{R}^d, \rho)$, a fact which we already in a way used in the proof of Lemma 2.1.23.

Lemma 2.1.33. *For $\gamma \in \mathbb{R}$, $p, q \in [1, \infty]$, a weight $\rho \in \boldsymbol{\rho}(\boldsymbol{\omega})$ and $k \in \mathbb{N}^d$ we have for $f \in \mathcal{B}^{\gamma}_{p,q,\mathfrak{s}}(\mathbb{R}^d, \rho)$*

$$\|\partial^k f\|_{\mathcal{B}^{\gamma-|k|_{\mathfrak{s}}}_{p,q,\mathfrak{s}}(\mathbb{R}^d,\rho)} \lesssim \|f\|_{\mathcal{B}^{\gamma}_{p,q,\mathfrak{s}}(\mathbb{R}^d,\rho)}$$

Proof. As in the beginning of the proof of Lemma 2.1.23 we use the estimate (2.25), which reads as

$$\|\Delta_j \partial^k f\|_{L^p(\mathbb{R}^d,\rho)} = \|\partial^k \Delta_j f\|_{L^p(\mathbb{R}^d,\rho)} \lesssim 2^{j|k|_{\mathfrak{s}}}\, \|\overline{\Delta}_j f\|_{L^p(\mathbb{R}^d,\rho)} .$$

with $\overline{\Delta}_j f = \sum_{i:\,|i-j|\leqslant 1} \Delta_i f$. From this inequality the claim easily follows. □

Paraproducts and multiplication in Besov spaces

Given two distributions $f_1, f_2 \in \mathcal{S}'_{\omega}(\mathbb{R}^d)$ we can use the Littlewood-Paley decomposition (2.19) to, at least formally, decompose their product in three terms

$$\begin{aligned} f_1 \cdot f_2 &= \sum_{j_1,j_2 \geqslant -1} \Delta_{j_1} f_1 \cdot \Delta_{j_2} f_2 \\ &= \sum_{j_2>0} S_{j_2-1} f_1 \cdot \Delta_{j_2} f_2 + \sum_{j_1,\, j_2:\, |j_1-j_2|\leqslant 1} \Delta_{j_1} f_1 \cdot \Delta_{j_2} f_2 + \sum_{j_1>0} \Delta_{j_1} f_1 \cdot S_{j_1-1} f_2 \\ &=: f_1 \prec f_2 + f_1 \circ f_2 + f_1 \succ f_2 ,\end{aligned}$$

which is known as the *Bony decomposition* [Bon81]. The objects $f_1 \prec f_2$ and $f_1 \succ f_2$ are known as *paraproducts*, while the term $f_1 \circ f_2$ is called the *resonance product.* Note that by definition

$$f_1 \succ f_2 = f_2 \prec f_1 .$$

The following estimates are some general version of the famous *paraproduct estimates* [BCD11, Theorem 2.82, 2.85] and were first proved in [Bon81]. The ones we present here are in way similar to those in [PT16b, Lemma 2.1] but we will also allow for weights in the set $\boldsymbol{\rho}(\boldsymbol{\omega})$ as above.

Lemma 2.1.34. *Given $\rho_1, \rho_2 \in \boldsymbol{\rho}(\omega)$ for $\omega \in \boldsymbol{\omega}$ and p, p_1, p_2, q, q_1, $q_2 \in [1, \infty]$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$ we have the bounds:*

- *For any $\gamma_2 \in \mathbb{R}$*

$$\|f_1 \prec f_2\|_{\mathcal{B}^{\gamma_2}_{p,q_2,\mathfrak{s}}(\mathbb{R}^d,\rho_1\cdot\rho_2)} \lesssim \|f_1\|_{L^{p_1}(\mathbb{R}^d,\rho_1,\mathfrak{s})} \|f_2\|_{\mathcal{B}^{\gamma_2}_{p_2,q_2,\mathfrak{s}}(\mathbb{R}^d,\rho_2)} .$$

- *For any $\gamma_1 < 0$, $\gamma_2 \in \mathbb{R}$*

$$\|f_1 \prec f_2\|_{\mathcal{B}^{\gamma_1+\gamma_2}_{p,q,\mathfrak{s}}(\mathbb{R}^d,\rho_1\cdot\rho_2)} \lesssim \|f_1\|_{\mathcal{B}^{\gamma_1}_{p_1,q_1,\mathfrak{s}}(\mathbb{R}^d,\rho_1)} \|f_2\|_{\mathcal{B}^{\gamma_2}_{p_2,q_2,\mathfrak{s}}(\mathbb{R}^d,\rho_2)} .$$

- *For any $\gamma_1, \gamma_2 \in \mathbb{R}$ with $\gamma_1 + \gamma_2 > 0$*

$$\|f_1 \circ f_2\|_{\mathcal{B}^{\gamma_1+\gamma_2}_{p,q,\mathfrak{s}}(\mathbb{R}^d,\rho_1\cdot\rho_2)} \lesssim \|f_1\|_{\mathcal{B}^{\gamma_1}_{p_1,q_1,\mathfrak{s}}(\mathbb{R}^d,\rho_1)} \|f_2\|_{\mathcal{B}^{\gamma_2}_{p_2,q_2,\mathfrak{s}}(\mathbb{R}^d,\rho_2)} .$$

Proof. We will write $(\epsilon^r_j)_{j\geqslant -1}$ for an object (that might change from line to line) such that $\|(\epsilon^r_j)_{j\geqslant -1}\|_{\ell^r} \leqslant 1$.

Note that the object $S_{j_2-1}f_1 \cdot \varDelta_{j_2} f_2$ has spectral support in in a rectangular annulus $2^{j_2\mathfrak{s}}\mathcal{A}$ so that with Lemma 2.1.12 we have

$$\varDelta_j \left(S_{j_2-1}f_1 \cdot \varDelta_{j_2} f_2\right) = \mathbf{1}_{j\sim j_2}\, \Psi^j * \left(S_{j_2} f_1 \cdot \varDelta_{j_2} f_2\right)$$

Application of (2.20) and Hölder's inequality then yields

$$\begin{aligned} \|\varDelta_j \left(S_{j_2-1}f_1 \cdot \varDelta_{j_2} f_2\right)\|_{L^p(\mathbb{R}^d,\rho_1\,\rho_2)} &\lesssim \mathbf{1}_{j\sim j_2} \|S_{j_2-1}f_1\|_{L^{p_1}(\mathbb{R}^d,\rho_1)} \|\varDelta_{j_2} f_2\|_{L^{p_2}(\mathbb{R}^d,\rho_2)} \\ &\lesssim \mathbf{1}_{j\sim j_2} \|S_{j_2-1}f_1\|_{L^{p_1}(\mathbb{R}^d,\rho_1)} \cdot \epsilon^{q_2}_j 2^{-j\gamma_2} \|f_2\|_{\mathcal{B}^{\gamma_2}_{p_2,q_2,\mathfrak{s}}(\mathbb{R}^d,\rho_2)} . \end{aligned}$$

Writing $S_{j_2-1}f_1 = \Phi^{<j_2-1} * f_1$ we obtain with Lemma 2.1.10 together with scaling of $\Psi^{<j_2-1}$ from Lemma 2.1.14 the estimate $\|S_{j_2-1}f_1\|_{L^{p_1}(\mathbb{R}^d,\rho_1)} \lesssim \|f_1\|_{L^{p_1}(\mathbb{R}^d,\rho_1)}$ and thus the first bound.

The second estimate follows as soon as we realize that by the same argument as in [BCD11, Proposition 2.33] $\|S_{j_2-1}f_1\|_{L^{p_1}(\mathbb{R}^d)} \lesssim \epsilon^{q_1}_{j_2} 2^{-j_2\gamma_2} \|f_1\|_{\mathcal{B}^{\gamma_1}_{p_1,q_1,\mathfrak{s}}(\mathbb{R}^d,\rho_1)}$. In total, again with (2.20),

$$\|\varDelta_j \left(S_{j_2-1}f_1 \cdot \varDelta_{j_2} f_2\right)\|_{L^p(\mathbb{R}^d,\rho_1\,\rho_2)} \lesssim \epsilon^{q_2}_j \epsilon^{q_1}_j\, 2^{-j_2(\gamma_1+\gamma_2)} \|f_1\|_{\mathcal{B}^{\gamma_1}_{p_1,q_1,\mathfrak{s}}(\mathbb{R}^d,\rho_1)} \|f_2\|_{\mathcal{B}^{\gamma_2}_{p_2,q_2,\mathfrak{s}}(\mathbb{R}^d,\rho_2)}$$

and the second estimate follows due to $\epsilon^{q_2}_j \epsilon^{q_1}_j \lesssim \epsilon^q_j$.

To show the estimate on the resonance term we note the spectral support of each term $\sum_{j_1\colon |j_1-j_2|\leqslant 1} \Delta_{j_1} f_1 \cdot \Delta_{j_2} f_2$ is contained in a box $2^{j_2\mathfrak{s}}\mathcal{B}$, so that

$$\Delta_j\Big(\sum_{j_1,j_2\colon |j_1-j_2|\leqslant 1} \Delta_{j_1} f_1 \cdot \Delta_{j_2} f_2\Big) = \sum_{j_1,j_2\gtrsim j\colon |j_1-j_2|\leqslant 1} \Delta_j\left(\Delta_{j_1} f_1 \cdot \Delta_{j_2} f_2\right).$$

Applying once more (2.20) we obtain

$$\begin{aligned}
&\Big\|\Delta_j\big(\sum_{j_1,j_2\colon |j_1-j_2|\leqslant 1} \Delta_{j_1} f_1 \Delta_{j_2} f_2\big)\Big\|_{L^p(\mathbb{R}^d,\rho_1\rho_2)} \\
&\lesssim \sum_{j_1,j_2\gtrsim j\colon |j_1-j_2|\leqslant 1} \|\Delta_{j_1} f_1\|_{L^{p_1}(\mathbb{R}^d,\rho_1)} \|\Delta_{j_2} f_2\|_{L^{p_2}(\mathbb{R}^d,\rho_2)} \\
&= 2^{-j(\gamma_1+\gamma_2)} \sum_{j_1,j_2\gtrsim j\colon |j_1-j_2|\leqslant 1} 2^{-(j_1-j)(\gamma_1+\gamma_2)} \\
&\cdot 2^{j_1\gamma_1}\|\Delta_{j_1} f_1\|_{L^{p_1}(\mathbb{R}^d,\rho_1)} 2^{j_2\gamma_2}\|\Delta_{j_2} f_2\|_{L^{p_2}(\mathbb{R}^d,\rho_2)} \cdot 2^{(j_1-j_2)\gamma_2} \\
&\lesssim 2^{-j(\gamma_1+\gamma_2)} \cdot \|f_1\|_{\mathcal{B}^{\gamma_1}_{p_1,q_1,\mathfrak{s}}(\mathbb{R}^d,\rho_1)} \|f_2\|_{\mathcal{B}^{\gamma_2}_{p_2,q_2,\mathfrak{s}}(\mathbb{R}^d,\rho_2)} \sum_{j_1\gtrsim j} 2^{-(j-j_1)(\gamma_1+\gamma_2)} \epsilon^q_{j_1}
\end{aligned}$$

where we used $\epsilon^q_{j_1} \lesssim \epsilon^{q_1}_{j_1} \cdot \epsilon^{q_2}_{j_2} \mathbf{1}_{|j_1-j_2|\leqslant 1}$. The last estimate now follows by application of Young's inequality for sequences on the sum over $j_1 \gtrsim j$. □

From these estimate one easily gets a rule how to mulitply distributions $f_1 \in \mathcal{B}^{\gamma_1}_{p_1,q_1,\mathfrak{s}}(\mathbb{R}^d,\rho_1)$ and $f_2 \in \mathcal{B}^{\gamma_2}_{p_2,q_2,\mathfrak{s}}(\mathbb{R}^d,\rho_2)$ provided one has the (canonical) requirement $\gamma_1 + \gamma_2 > 0$.

Corollary 2.1.35. *Let $\rho,\rho_1,\rho_2 \in \boldsymbol{\rho}(\omega)$ with $\rho = \rho_1 \cdot \rho_2$ and p, p_1, p_2, q, q_1, $q_2 \in [1,\infty]$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$. For γ, $\gamma_1, \gamma_2 \in \mathbb{R}\backslash\{0\}$ with $\gamma_1+\gamma_2 > 0$ and $\gamma = \gamma_1 \wedge \gamma_2 \wedge (\gamma_1+\gamma_2)$ one has the bound*

$$\|f_1 \cdot f_2\|_{\mathcal{B}^{\gamma}_{p,\tilde{q},\mathfrak{s}}(\mathbb{R}^d)} \lesssim \|f_1\|_{\mathcal{B}^{\gamma_1}_{p_1,q_1}(\mathbb{R}^d,\rho_1)} \cdot \|f_2\|_{\mathcal{B}^{\gamma_2}_{p_2,q_2,\mathfrak{s}}(\mathbb{R}^d,\rho_2)}, \tag{2.33}$$

where $\tilde{q} = q_1 \vee q_2$ and further for any $\kappa > 0$

$$\|f_1 \cdot f_2\|_{\mathcal{B}^{\gamma-\kappa}_{p,q,\mathfrak{s}}(\mathbb{R}^d)} \lesssim \|f_1\|_{\mathcal{B}^{\gamma_1}_{p_1,q_1}(\mathbb{R}^d,\rho_1)} \cdot \|f_2\|_{\mathcal{B}^{\gamma_2}_{p_2,q_2,\mathfrak{s}}(\mathbb{R}^d,\rho_2)}. \tag{2.34}$$

Remark 2.1.36. *One has to say of course what we mean by a product $f_1 \cdot f_2$ for distributional f_1, f_2. We here simply* define *it to be the sum of*

$$f_1 \cdot f_2 := f_1 \prec f_2 + f_1 \circ f_2 + f_1 \succ f_2$$

where every term on the right hand side is well-defined due to $\gamma_1+\gamma_2 > 0$ and Lemma 2.1.34.

Proof. Note first that $\tilde{q} \geqslant q$ so that we have $\tilde{q} = q_1 \vee q_2 \vee q$. We show (2.33), the bound (2.34) follows then by an application of Lemma 2.1.27. We split

$$f_1 \cdot f_2 = f_1 \prec f_2 + f_2 \prec f_1 + f_1 \circ f_2$$

and bound every term on the right hand seperately. Let's first consider the case $\gamma_1, \gamma_2 > 0$. In this case we bound via Lemma 2.1.26 and 2.1.34

$$\begin{aligned}\|f_1 \prec f_2\|_{\mathcal{B}^{\gamma}_{p,\tilde{q},\mathfrak{s}}(\mathbb{R}^d,\rho)} &\lesssim \|f_1 \prec f_2\|_{\mathcal{B}^{\gamma_2}_{p,q_2,\mathfrak{s}}(\mathbb{R}^d,\rho)} \lesssim \|f_1\|_{L^{p_1}(\mathbb{R}^d,\rho_1)} \|f_2\|_{\mathcal{B}^{\gamma_2}_{p_2,q_2,\mathfrak{s}}(\mathbb{R}^d,\rho_2)} \\ &\lesssim \|f_1\|_{\mathcal{B}^{\gamma_1}_{p_1,q_1}(\mathbb{R}^d,\rho_1)} \|f_2\|_{\mathcal{B}^{\gamma_2}_{p_2,q_2,\mathfrak{s}}(\mathbb{R}^d,\rho_2)} ,\end{aligned}$$

where we used Lemma 2.1.26 and 2.1.34 in the first and Lemma 2.1.28 in the second line. The term $f_2 \prec f_1$ can be handled in the exact same manner. For the resonance product we obtain with Lemma 2.1.26 and Lemma 2.1.34

$$\|f_1 \circ f_2\|_{\mathcal{B}^{\gamma}_{p,\tilde{q},\mathfrak{s}}(\mathbb{R}^d,\rho)} \lesssim \|f_1 \circ f_2\|_{\mathcal{B}^{\gamma_1+\gamma_2}_{p,q,\mathfrak{s}}(\mathbb{R}^d,\rho)} \lesssim \|f_1\|_{\mathcal{B}^{\gamma_1}_{p_1,q_1}(\mathbb{R}^d,\rho_1)} \|f_2\|_{\mathcal{B}^{\gamma_2}_{p_2,q_2,\mathfrak{s}}(\mathbb{R}^d,\rho_2)} .$$

It remains to consider the case where one of the γ_1, γ_2 is negative. We assume without loss of generality $\gamma_1 < 0 < \gamma_2$, in which case we estimate with Lemma 2.1.26 and 2.1.34

$$\|f_1 \prec f_2\|_{\mathcal{B}^{\gamma}_{p,\tilde{q},\mathfrak{s}}(\mathbb{R}^d,\rho)} \lesssim \|f_1 \prec f_2\|_{\mathcal{B}^{\gamma_1+\gamma_2}_{p,q,\mathfrak{s}}(\mathbb{R}^d,\rho)} \lesssim \|f_1\|_{\mathcal{B}^{\gamma_1}_{p_1,q_1}(\mathbb{R}^d,\rho_1)} \cdot \|f_2\|_{\mathcal{B}^{\gamma_2}_{p_2,q_2,\mathfrak{s}}(\mathbb{R}^d,\rho_2)} ,$$

the remaining terms $f_2 \prec f_1$ and $f_1 \circ f_2$ can be bounded as above. □

2.2 White noise

All the stochastics in this thesis origins in the following object.

Definition 2.2.1. *A random distribution $\xi \in \mathcal{S}'(\mathbb{R}^d)$ is called* white noise *on $\mathbb{R}^d$, if for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$*

$$\xi(\varphi) \overset{\mathrm{d}}{\sim} \mathcal{N}\left(0, \|\varphi\|^2_{L^2(\mathbb{R}^d)}\right) ,$$

where $\mathcal{N}(0, \sigma^2)$ denotes the normal distribution.

The existence of white noise is proved by the Bochner-Minlos theorem (which is in fact true on a far broader setup, see for example [Oba94, Theorem 1.5.2]). Similar one can construct periodic white noise on a torus $\mathbb{T}^d = [-1/2, 1/2)^d$ by requiring instead $\xi(\varphi) \sim \mathcal{N}(0, \|\varphi\|_{L^2(\mathbb{T}^d)})$. It will be of some importance to recall the smoothness and decay of ξ, which is the content of the following Lemma.

Lemma 2.2.2. *Let ξ be white noise on $\mathbb{R}^d$ and $\mathfrak{s} \in [1,\infty)^d$ some scaling vector as in Section 2.1. It holds for any $\gamma < -|\mathfrak{s}|/2$, $\delta > 0$ and $p \in [1,\infty)$*

$$\mathbb{E}\left[\|\xi\|^p_{\mathcal{C}^\gamma_{\mathfrak{s}}(\mathbb{R}^d,\langle x\rangle^{-\delta})}\right] < \infty$$

with $\langle x\rangle^{-\delta} := (1+|x|^2)^{-\delta/2}$ as in (2.3).

Remark 2.2.3. *Note, that $|x|$ here really denotes the classical,* isotropic *euclidean distance and not the scaled norm introduced in* (2.12).

Proof. Considering the Besov embedding in Lemma 2.1.26 it is actually enough to show instead for (large) $p \in [1,\infty)$

$$\mathbb{E}\left[\|\xi\|^p_{\mathcal{B}^\gamma_{p,p,\mathfrak{s}}(\mathbb{R}^d,\langle x\rangle^{-\delta})}\right] < \infty\,.$$

Without loss of generality we choose p large enough such that $p \cdot \delta > d$. We then have

$$\begin{aligned}\mathbb{E}\left[\|\xi\|^p_{\mathcal{B}^\gamma_{p,p,\mathfrak{s}}(\mathbb{R}^d,\langle x\rangle^{-\delta})}\right] &= \sum_{j\geqslant -1} 2^{j\gamma p}\int_{\mathbb{R}^d}\mathrm{d}x\,\langle x\rangle^{-p\delta}\,\mathbb{E}[|(\Delta_j\xi)_x|^p]\\ \lesssim \sum_{j\geqslant -1} 2^{j\gamma p}\sup_{x\in\mathbb{R}^d}\mathbb{E}[|(\Delta_j\xi)_x|^p]\,.\end{aligned}$$

Using equivalence of moments for the Gaussian random variable $\Delta_j\xi = \Psi^j * \xi$ and the Definition of white noise we can bound the expectation on the right hand side by

$$\mathbb{E}[|(\Delta_j\xi)_x|^p] \lesssim \mathbb{E}[|(\Delta_j\xi)_x|^2]^{p/2} \lesssim \|\Psi^j\|^p_{L^2(\mathbb{R}^d)} \lesssim 2^{-j\frac{|\mathfrak{s}|}{2}p}\,,$$

where we applied Lemma 2.1.10 together with the scaling of Ψ^j from Lemma 2.1.14 to deduce $\|\Psi^j\|_{L^2(\mathbb{R}^d)} \lesssim 2^{-j\frac{|\mathfrak{s}|}{2}}$. The claim follows. □

The polynomial weight on ξ is actually far too strong as the right growth turns out to be $\sqrt{\log(x)}$. This follows in particular from the following lemma, which we take from [AC15, Lemma 5.3]. For simplicity we only consider the isotropic scaling case $\mathfrak{s} = (1,\dots,1)$, which will be all we need.

Lemma 2.2.4. *Let ξ be white noise on $\mathbb{R}^d$ and fix uniformly bounded $\chi_k \in C^{d+1}(\mathbb{R}^2)$ with* $\operatorname{supp}\chi_k \subseteq [-k-1,k+1]^2$ *and $\chi_k|_{[-k,k]^2} = 1$. For $\gamma < -\frac{d}{2}$ there are $\lambda,\lambda' > 0$ such that*

$$\sup_{k\in\mathbb{N}}\frac{\mathbb{E}\left[\exp(\lambda\|\xi\chi_k\|^2_{C^\gamma(\mathbb{R}^d)})\right]}{k^{\lambda'}} < \infty\,.$$

Remark 2.2.5. *In [AC15] the authors actually bound periodic white noise ξ_k on $[-k,k)^2$ instead of $\chi_k\xi$. However, this can be easily translated to the result above due to $\xi_{k+1}\chi_k \stackrel{\mathrm{d}}{=} \xi\chi_k$ and $\|\chi_k\xi_{k+1}\|_{\mathcal{C}^\gamma(\mathbb{R}^2)} \lesssim \|\xi_{k+1}\|_{\mathcal{C}^\gamma([-k-1,k+1)^2)}$ for $\gamma \in (0,1)$. Although the proof in [AC15] is in $d = 2$, it can be readily translated to general dimensions.*

It should further cause no problem to reformulate the lemma for general, anisotropic scaling $\mathfrak{s}$, but if we intend to just copy the proof from [AC15] we would need periodic, anisotropic Besov spaces $\mathcal{B}^\gamma_{p,q,\mathfrak{s}}(\mathbb{T}^d)$ in this case, which we did not define, so that we refrain from doing so.

2.3 The theory of regularity structures

In the following we fix once more a scaling vector

$$\mathfrak{s} = (\mathfrak{s}_1, \ldots, \mathfrak{s}_d) \in [1,\infty)^d \,.$$

We allow $\mathfrak{s}$ to be *fractional*, that is with possible non-integer components $\mathfrak{s}_i$, in contrast to [Hai14]. The usage of a fractional scaling means of course that one actually would have to recheck all the results in [Hai14]. However, the only deep result we really use from [Hai14] is the Reconstruction Theorem 2.3.19 below, which can be readily repeated for a fractional scaling by using a fractionaly scaled wavelet basis instead, as introduced for example in [Tri06].

For $\varphi \in L^1(\mathbb{R}^d)$ we sometimes need the notation φ^λ by which we mean the L^1 scaling

$$\varphi^\lambda := \lambda^{-|\mathfrak{s}|}\varphi(\lambda^{-\mathfrak{s}}\cdot)\,.$$

so that φ^λ_x should be read as

$$\varphi^\lambda_x = \lambda^{-|\mathfrak{s}|}\varphi(\lambda^{-\mathfrak{s}}x)\,. \tag{2.35}$$

Remark 2.3.1. ⚠*Note that we slightly differ here from [Hai14], where this notation denotes the function $z \mapsto \lambda^{-|\mathfrak{s}|}\varphi(\lambda^{-\mathfrak{s}}(z-x))$ instead.*

Let's start by recalling the definition of a regularity structure.

Definition 2.3.2. *[Hai14, Definition 2.1] A regularity structure is a triple $\mathscr{T} = (A, \mathcal{T}, G)$ consisting of:*

- *A locally finite* index set *$A \subseteq \mathbb{R}$, $0 \in A$, bounded from below.*
- *A* model space *$\mathcal{T} = \bigoplus_{\alpha\in A}\mathcal{T}_\alpha$ where each $\mathcal{T}_\alpha$ is a Banach space equipped with a norm $\|\cdot\|_{\mathcal{T}_\alpha}$. $\mathcal{T}_0$ is spanned by a unit vector which we call $\mathbf{1}$.*

- A structure group G *of linear operators acting on* $\mathcal{T}$ *such that for every* $\Gamma \in G$, $\alpha \in A$, $\tau \in \mathcal{T}_\alpha$

$$\Gamma a - a \in \bigoplus_{\beta<\alpha} \mathcal{T}_\beta \,.$$

The elements of A *are called* homogeneities.

Remark 2.3.3. *Usually the space* $\mathcal{T}$ *is taken as a real space, which we will also be the standard used here. However, one might also choose the field* $\mathbb{C}$ *if necessary.*

Let us also recall the following two notions from [Hai14]

Definition 2.3.4. *Given a regularity structure* $\mathscr{T} = (A, \mathcal{T}, G)$ *we say that* $\mathcal{V} = \bigoplus_{\alpha\in A} \mathcal{V}_\alpha \subseteq \mathcal{T}$ *is a sector (of regularity* β*) of* $\mathscr{T}$ *if every* $\mathcal{V}_\alpha$ *for* $\alpha \in A$ *is a subspace of* $\mathcal{T}_\alpha$ *and has a complement* $\mathcal{T}_\alpha \backslash \mathcal{V}_\alpha$ *such that* $\mathcal{V}_\alpha \oplus \mathcal{T}_\alpha \backslash \mathcal{V}_\alpha = \mathcal{T}_\alpha$ *and if one has* $G(\mathcal{V}) \subseteq \mathcal{V}$ *(and* $\mathcal{V}_\alpha = \{0\}$ *for* $\alpha < \beta$*). We will write* $A_\mathcal{V} \subseteq A$ *for the set of those* $\alpha \in A$ *such that* $\mathcal{V}_\alpha \neq \{0\}$*. We call a sector* $\mathcal{V}$ *of regularity* 0 function like.

Given two regularity structures $\mathscr{T} = (A, \mathcal{T}, G)$, $\mathscr{T}' = (A', \mathcal{T}', G')$ *we write* $\mathscr{T} \subseteq \mathscr{T}'$ *if* $A \subseteq A'$ *and if there is a linear and continuous injection map* $\imath : \mathcal{T} \to \mathcal{T}'$ *such that* $\imath(\mathcal{T}_\alpha) \subseteq \mathcal{T}'_\alpha$ *for every* $\alpha \in a$ *and*

$$G'\imath\mathcal{T} \subseteq \imath\mathcal{T},\ \imath^{-1}G'\imath = G \,.$$

The continuity of $\imath$ *should be read in the sense that for every* $\alpha \in A$ *the restriction* $\imath|_{\mathcal{T}_\alpha} : \mathcal{T}_\alpha \to \mathcal{T}'_\alpha$ *is continuous.*

Remark 2.3.5. *A sector* $\mathcal{V} \subseteq \mathcal{T}$ *is again a regularity structure up to the requirement* $0 \in A$, $\mathbf{1} \in \mathcal{T}_0$ *so that the inclusion* $\mathcal{V} \subseteq \mathcal{T}$ *can also be read in the sense of regularity structures with the trivial embedding. In this sense one can define for instance a* subsector $\mathcal{W} \subseteq \mathcal{V}$.

Remark 2.3.6. *We will usually directly identify* $\mathcal{T}$ *with its image under* $\imath$ *so that* $\imath$ *can be replaced by the trivial embedding and is not mentioned any further.*

It turns out that in many cases in this thesis the right space to work on is the following object.

Definition 2.3.7. *Given a regularity structure* $\mathcal{T}$ *and two sectors* $\mathcal{W}$, $\mathcal{V}$ *such that* $\mathcal{W} \subseteq \mathcal{V}$ *in the sense of Remark 2.3.5 we say that* $\mathcal{V}\backslash\mathcal{W} = \bigoplus_{\alpha\in A}(\mathcal{V}\backslash\mathcal{W})_\alpha \subseteq \mathcal{V}$ *is the* complement *of the sector* $\mathcal{W}$ *within* $\mathcal{V}$ *if for any* $\alpha \in A$ *one has* $\mathcal{W}_\alpha \oplus (\mathcal{V}\backslash\mathcal{W})_\alpha = \mathcal{V}_\alpha$ *and if* $(\mathcal{V}\backslash\mathcal{W})_\alpha = \{0\}$ *whenever* $\alpha \in A_\mathcal{W}$ *(so that either* $\mathcal{W}_\alpha$ *or* $(\mathcal{V}\backslash\mathcal{W})_\alpha$ *is trivial).*

We will write $A_{\mathcal{V}\backslash\mathcal{W}}$ *for those* $\alpha \in A$ *for which* $(\mathcal{V}\backslash\mathcal{W})_\alpha \neq \{0\}$ *(in particular* $A_W \uplus A_{\mathcal{V}\backslash\mathcal{W}} = A_\mathcal{V}$*). We call the minimal element of* $A_{\mathcal{V}\backslash\mathcal{W}}$ *the regularity of* $\mathcal{V}\backslash\mathcal{W}$*.*

Remark 2.3.8. *The condition* $(\mathcal{V}\backslash\mathcal{W})_\alpha = \mathcal{V}_\alpha \Leftrightarrow (\mathcal{V}\backslash\mathcal{W})_\alpha \neq \{0\}$ *for* $\alpha \in A_{\mathcal{V}}$ *we imposed in Definition 2.3.7 seems quite restrictive and is in fact not really needed for the statements in this thesis, but it greatly simplify the notations and is completely sufficient for our purposes (where complements of sectors will always be of the form* (2.45) *below).*

For $\tau \in \mathcal{T}$ and $\alpha \in A$ we write τ^α for the projection of τ on $\mathcal{T}_\alpha$. If $\dim \mathcal{T}_\alpha < \infty$ and we have a basis $\{e_i\}$ for $\mathcal{T}_\alpha$ we then also write τ^{e_j} for the coefficient of τ^α with respect to e_j. For example if $\tau \in \mathcal{T}$ with $\tau - c \cdot \mathbf{1} \in \bigoplus_{\alpha\in A,\,\alpha\neq 0} \mathcal{T}_\alpha$ we have $\tau^{\mathbf{1}} = c$. Note that by this definition $\tau^\alpha \in \mathcal{T}_\alpha$ is a vector in a Banach space, while τ^{e_i} is a real (or complex) number. We may also write

$$\tau^{<\gamma} := \sum_{\alpha\in A\colon \alpha<\gamma} \tau^\alpha$$

Similarly we define $\tau^{>\gamma}$. For $\Gamma \in G$ we use the abbreviation

$$\Gamma^\alpha \tau := (\Gamma\tau)^\alpha \tag{2.36}$$

and proceed similar for other operators acting $\mathcal{T}$. The same remark applies for the "basis notation" above so that for instance $\Gamma^{\mathbf{1}}\tau := (\Gamma\tau)^{\mathbf{1}}$.

We will also need the object

$$\mathcal{T}_\gamma^- := \bigoplus_{\alpha\in A\colon \alpha<\gamma} \mathcal{T}_\alpha$$

for $\gamma \in \mathbb{R}$, so that for example $\tau^{<\gamma} \in \mathcal{T}_\gamma^-$ for $\tau \in \mathcal{T}$. We use the same notations for sectors (and for complements of sectors which we introduce in Definition 2.3.7 below). Let us now introduce the notion of a model.

Definition 2.3.9. *A model for a regularity structure* $\mathscr{T}$ *is a family of linear maps* $\Gamma_{xy} \in G$, $\Pi_x : \mathcal{T} \to \mathcal{S}'(\mathbb{R}^d)$ *for* $x, y \in \mathbb{R}^d$ *that satisfy for* $x, y, z \in \mathbb{R}^d$

$$\Gamma_{xx} = \mathrm{Id}_{\mathcal{T}},\ \Gamma_{xy}\Gamma_{yz} = \Gamma_{xz},\ \Pi_x = \Pi_y\Gamma_{yx} \tag{2.37}$$

and further for $\alpha, \beta \in A$, $\tau \in \mathcal{T}_\alpha$ *and* $\beta < \alpha$

$$\|\Gamma_{yx}^\beta \tau\|_{\mathcal{T}_\beta} \lesssim \|\tau\|_{\mathcal{T}_\alpha} \cdot \|x - y\|_{\mathfrak{s}}^{\alpha-\beta}, \tag{2.38}$$

$$|\Pi_x\tau(\varphi_{\cdot-x}^\lambda)| \lesssim \|\tau\|_{\mathcal{T}_\alpha} \cdot \lambda^\alpha\,, \tag{2.39}$$

with $\varphi_{\cdot-x}^\lambda = \lambda^{-|\mathfrak{s}|}\varphi(\lambda^{-\mathfrak{s}}(\cdot - x))$, *uniformly over all* $\lambda \in (0,1]$ *and* $\varphi \in C_c^\infty(\mathbb{R}^d)$, $\mathrm{supp}\,\varphi \subseteq B_{\mathfrak{s}}(0,1)$, $\|\varphi\|_{C^r} \leqslant 1$, *with* $r \in \mathbb{N}$ *being the smallest number strictly bigger than* $-\min A$. *As in* (2.36) *we wrote* $\Gamma_{yx}^\beta\tau := (\Gamma_{yx}\tau)^\beta$ *for the projection of* $\Gamma_{yx}\tau$ *onto* $\mathcal{T}_\beta$.

We further introduce:

$$\begin{aligned}
\|\Pi\|_\gamma &:= \sup_{\varphi} \sup_{\alpha\in A,\, \alpha<\gamma} \sup_{\tau\in\mathcal{T}_\alpha,\, \|\tau\|_{\mathcal{T}_\alpha}\leqslant 1} \sup_{\lambda\in(0,1]} \lambda^{-\alpha}|\Pi_x\tau(\varphi^\lambda_{\cdot-x})|\,,\\
\|\Gamma\|_\gamma &:= \sup_{x,\,y\in\mathbb{R}^d,\, x\neq y} \sup_{\alpha,\beta\in A,\, \beta<\alpha<\gamma,\, \tau\in\mathcal{T}_\alpha,\, \|\tau\|_{\mathcal{T}_\alpha}\leqslant 1} \|\Gamma^\beta_{yx}\tau\|_{\mathcal{T}_\beta}\|x-y\|_{\mathfrak{s}}^{\beta-\alpha}\,,\\
\vert\!\vert\!\vert(\Pi,\Gamma)\vert\!\vert\!\vert_\gamma &:= \|\Pi\|_\gamma + \|\Gamma\|_\gamma\,,
\end{aligned}$$

where $\sup_\varphi$ *runs over the class of* φ *described above. Given two models* (Π,Γ), $(\hat\Pi,\hat\Gamma)$ *we also defines "distances" by* $\vert\!\vert\!\vert(\Pi,\Gamma);(\hat\Pi,\hat\Gamma)\vert\!\vert\!\vert_\gamma := \vert\!\vert\!\vert(\Pi-\hat\Pi,\Gamma-\hat\Gamma)\vert\!\vert\!\vert_\gamma = \|\Pi-\hat\Pi\|_\gamma + \|\Gamma-\hat\Gamma\|_\gamma$ *(with a slight abuse of notation since* $(\Pi-\hat\Pi,\Gamma-\hat\Gamma)$ *is in general not a model).*

We may write $\Gamma_{y,x}$ *instead of* Γ_{yx} *to separate the arguments more clearly.*

Note that we require global bounds on the model (Π,Γ) in Definition 2.3.9, which is in contrast to [Hai14] where the corresponding bounds do only need to hold on compact sets. However, we will sometimes also encounter the case where the norms above only hold for x,y in some set $\Omega\subseteq\mathbb{R}^d$, in which case we write instead

$$\|\Pi\|_{\gamma,\Omega},\ \|\Gamma\|_{\gamma,\Omega},\ \vert\!\vert\!\vert(\Pi,\Gamma)\vert\!\vert\!\vert_{\gamma,\Omega},\ \vert\!\vert\!\vert(\Pi,\Gamma);(\hat\Pi,\hat\Gamma)\vert\!\vert\!\vert_{\gamma,\Omega} \tag{2.40}$$

in accordance with [Hai14, Definition 2.17]. If Π and Γ are as in Definition 2.3.9 but only with local bounds $\|\Pi\|_{\gamma,\mathfrak{K}}$, $\|\Gamma\|_{\gamma,\mathfrak{K}}<\infty$ for any compact set $\mathfrak{K}\subseteq\mathbb{R}^d$, i.e. if (Π,Γ) is a model in the sense of [Hai14, Definition 2.17], we will say in this thesis that (Π,Γ) is a *model with local bounds.*

The main reason for requiring global estimates is that we work in Chapter 5 and 6 with an approach based on Fourier analysis, for which it seems unavoidable to work with bounds on the full space. Compare also [HL17] for another work with these assumptions. Global bounds case are given immediately in the case if the considered SPDE has periodic white noise in space. In case of space-time white noise which is periodic in space one can choose noise which is periodic in time as well, with a period bigger than the considered time horizon of the equation. If one wants to consider problems with non-periodic noise, one would have to introduce weights as in Chapter 3, 4 and 8 of this thesis. We will avoid doing so for the sake of simplicity.

We will frequently use the following lemma from [GIP15, Lemma 6.3] [4]

Lemma 2.3.10. *Given a regularity structure* $\mathscr{T}$ *with model* (Π,Γ) *it holds for any* $\gamma>0, A\ni\alpha<\gamma$, $\tau\in\mathcal{T}_\alpha$, $\lambda\in(0,1]$, $x\in\mathbb{R}^d$ *and* $\varphi\in\mathcal{S}(\mathbb{R}^d)$

$$|\Pi_x\tau(\varphi^\lambda_{\cdot-x})| \lesssim (1+\|\Gamma\|_\gamma)\|\Pi\|_\gamma\cdot[\varphi]\cdot\lambda^\alpha\,,$$

[4] In [GIP15, Lemma 6.3] the authors consider only the isotropic scaling $\mathfrak{s}=(1,\dots,1)$, the translation to the general case is however immediate.

where $[\varphi] := \sup_{|\mu|_\mathfrak{s}\leqslant r,\, x\in\mathbb{R}^d}(1+\|x\|_\mathfrak{s})^{d+\gamma+r}|\partial^\mu\varphi(x)|$ *with* $r\in\mathbb{N}$ *being the smallest number strictly larger than* $-\min A$.

The functions $\Psi^j = \mathcal{F}^{-1}_{\mathbb{R}^d}\varphi_j$ introduced in Section 2.1 above where used as some sort of building blocks to decompose distributions f into smooth functions $\Delta_j f$. It is therefore not suprising that we can express bounds like (2.39) in terms of Ψ^j.

Lemma 2.3.11. *Given a regularity structure* $\mathscr{T} = (A,\mathcal{T},G)$ *with a model* (Π,Γ) *we have for* $\gamma\in\mathbb{R}$

$$\|\Pi\|_\gamma \approx \sup_{x\in\mathbb{R}^d,\, j\geqslant 0}\ \sup_{\substack{\alpha\in A,\,\alpha<\gamma\\ \tau\in T_\alpha,\,\|\tau\|_{\mathcal{T}_\alpha}=1}} 2^{j\alpha}|\Pi_x\tau(\Psi^{<j}_{\cdot-x})|\,,$$

uniformly for $\|\Gamma\|_\gamma$ *contained in a bounded set.*

Proof. The inequality "$\gtrsim$" follows from Lemma 2.3.10 and the scaling property $\Psi^{<j-1} = 2^{j|\mathfrak{s}|}\phi(2^{j\mathfrak{s}}\cdot)$ for some $\phi\in\mathcal{S}(\mathbb{R}^d)$ which we stated in Lemma 2.1.14. To show "$\lesssim$" assume that the right hand side, which we denote by

$$C_\Pi = \sup_{\substack{\alpha\in A,\,\alpha<\gamma\\ \tau\in T_\alpha,\,\|\tau\|_{\mathcal{T}_\alpha}=1}} 2^{j\alpha}|\Pi_x\tau(\Psi^{<j}_{\cdot-x})|\,,$$

is finite. Note that we have for $j\geqslant 0$ and $\tau\in\mathcal{T}_\alpha$ with $\alpha<\gamma$

$$|\Pi_x(\Psi^j_{x-u})| \lesssim C_\Pi 2^{-j\alpha} \tag{2.41}$$

since we can rewrite the test-function by $\Psi^j = \Psi^{<j+1}-\Psi^{<j}$ and then apply the triangle inequality. Let $\varphi\in C^r$ with r as in Definition 2.3.9 and with support contained in $B_\mathfrak{s}(0,1)$ and with $\|\varphi\|_{C^r}\leqslant 1$ and fix $\lambda\in(0,1]$. Choose $j'\geqslant 0$ such that $2^{-j'-1}<\lambda\leqslant 2^{-j'}$. We decompose for $\tau\in\mathcal{T}_\alpha$ with $\alpha<\gamma$

$$\Pi_x\tau(\varphi^\lambda_{\cdot-x}) = \Pi_x\tau\Big(\sum_{i\leqslant j'}\Delta_i\varphi^\lambda_{\cdot-x}\Big) + \sum_{i>j'}\Pi_x\tau(\Delta_i\varphi^\lambda_{\cdot-x})\,. \tag{2.42}$$

The first term can be bounded if we reshape it as follows:

$$\begin{aligned}
\Big|\Pi_x\tau\Big(\sum_{i\leqslant j'}\Delta_i\varphi^\lambda_{\cdot-x}\Big)\Big| &= \Big|\int \Pi_x\tau(\mathrm{d}u)\,(\Psi^{<j'+1}*\varphi^\lambda_{\cdot-x})(u)\Big|\\
&\overset{(\leftrightarrow)}{=} \Big|\int \mathrm{d}v\,\varphi^\lambda_{v-x}\int \Pi_x\tau(\mathrm{d}u)\,\Psi^{<j'+1}_{u-v}\Big|\\
&= \Big|\sum_{\alpha'\in A:\,\alpha'\leqslant\alpha}\int \mathrm{d}v\,\varphi^\lambda_{v-x}\Pi_v\Gamma^{\alpha'}_{vx}\tau(\Psi^{<j'+1}_{\cdot-v})\Big|\\
&\lesssim (1+\|\Gamma\|_\gamma)C_\Pi\sum_{\alpha'\in A:\,\alpha'\leqslant\alpha}\int \mathrm{d}v\,\varphi^\lambda_{v-x}\|v-x\|_\mathfrak{s}^{\alpha-\alpha'}\,2^{-(j'+1)\alpha'}\\
&\lesssim (1+\|\Gamma\|_\gamma)C_\Pi\sum_{\alpha'\in A:\,\alpha'\leqslant\alpha}\lambda^{\alpha-\alpha'}2^{-j'\alpha'} \lesssim (1+\|\Gamma\|_\gamma)C_\Pi\lambda^\alpha\,,
\end{aligned}$$

where we applied Definition 2.3.9 in the third line as well as the definition of C_Π. For the fourth line we used the scaling of φ^λ to substitute and replaced $2^{(j'+1)\alpha'} \lesssim 2^{j'\alpha'}$ since $\alpha' \leqslant \alpha < \gamma$ is contained in a bounded set. To justify the "Fubini like" step ($\leftrightarrow$) one can approximate $\Pi_x\tau$ by an integrable function and then pass to the limit on both sides.

It remains to bound the second term on the right hand side of (2.42).

Let $i > j' \geqslant 0$ and note that by spectral support properties, as in the proof of Lemma 2.1.23, we have the identity $\varDelta_i = \varDelta_i\overline{\varDelta}_i$ where $\overline{\varDelta}_i := \sum_{|i-j|\leqslant 1}\varDelta_j$. We denote by $\overline{\Psi}^i := \sum_{|j-i|\leqslant 1}\Psi^j$ the kernel belonging to $\overline{\varDelta}_i$. The function $\overline{\Psi}^i$, $i > 0$ (and Ψ^i) integrates polynomials to 0 since it is spectrally supported on a rectangular annulus away from 0 (compare Lemma 2.1.14). Choosing some $\sigma \in (-\min A, r)$ with $\sigma \notin |\mathbb{N}^d|_\mathfrak{s}$ and $r \in \mathbb{N}$ as in Definition 2.3.9 we rewrite, using the Taylor remainder from (2.22),

$$\begin{aligned}\Pi_x\tau(\varDelta_i\varphi^\lambda_{\cdot-x}) = \Pi_x\tau(\varDelta_i\overline{\varDelta}_i\varphi^\lambda_{\cdot-x}) &= \iint \mathrm{d}u_1\mathrm{d}u_2\,\overline{\Psi}^i_{u_1-u_2}\varphi^\lambda_{u_2}\,\Pi_x\tau(\Psi^i_{\cdot-u_1-x}))\\ &= \sum_{\alpha'\colon \alpha'\leqslant\alpha}\iint \mathrm{d}u_1\mathrm{d}u_2\,\overline{\Psi}^i_{u_1-u_2}\,R^\sigma_{u_1;u_2-u_1}\varphi^\lambda\cdot\Pi_{u_1+x}\Gamma^{\alpha'}_{u_1+x,x}\tau(\Psi^i_{\cdot-(u_1+x)})\,.\end{aligned}$$

We used in the second line that the polynomial in u_2 given by $T^\sigma_{u_1;u_2-u_1}\varphi^\lambda$ is integrated to 0. We can estimate this expression up to a constant using (2.41) and Lemma 2.1.20 by

$$\begin{aligned}&C_\Pi(1+\|\Gamma\|_\gamma)\sum_{\alpha'\in A\colon\alpha'\leqslant\alpha}2^{-i\alpha'}\sum_{k\in\mathbb{N}^d_{>\sigma}}\\ &\quad\times\iint\mathrm{d}u_1\mathrm{d}u_2\,|\overline{\Psi}^i_{u_1-u_2}|\,\|u_1-u_2\|_\mathfrak{s}^{|k|_\mathfrak{s}}\int_0^1\mathrm{d}t\,|\partial^k\varphi^\lambda_{u_1+v^k_t(u_2-u_1)}|\,\|u_1\|_\mathfrak{s}^{\alpha-\alpha'}\,,\end{aligned}$$

where $\mathbb{N}^d_{>\sigma} = \{k\in\mathbb{N}^d \mid |k|_\mathfrak{s} > \sigma,\ |k-e_{\mathfrak{m}(k)}|_\mathfrak{s} < \sigma\}$ is defined as in Lemma 2.1.20. Note that $|k-e_{\mathfrak{m}(k)}| \leqslant |k-e_{\mathfrak{m}(k)}|_\mathfrak{s} < \sigma$ implies $|k| < \sigma+1$ and thus since $|k|$ is an integer and $\sigma < r$ we must have $|k| \leqslant r$. Consequently the derivatives of φ in this expression can be uniformly bounded by $\|\varphi\|_{C^r} \leqslant 1$. Using that $\|v^k_t(u_2-u_1)\|_\mathfrak{s} \leqslant \|u_2-u_1\|_\mathfrak{s}$ we get $\|u_1\|_\mathfrak{s}^{\alpha-\alpha'} \lesssim \|u_1+v^k_t(u_2-u_1)\|_\mathfrak{s}^{\alpha-\alpha'} + \|u_2-u_1\|_\mathfrak{s}^{\alpha-\alpha'}$, which yields for the term above the bound

$$\begin{aligned}&C_\Pi(1+\|\Gamma\|_\gamma)\sum_{\alpha'\in A\colon\alpha'\leqslant\alpha}2^{-i\alpha'}\sum_{k\in\mathbb{N}^d_{>\sigma}}\lambda^{-|k|_\mathfrak{s}}\,2^{-i|k|_\mathfrak{s}}\,(\lambda^{\alpha-\alpha'}+2^{-i(\alpha-\alpha')})\\ &\lesssim C_\Pi(1+\|\Gamma\|_\gamma)\sum_{\alpha'\in A\colon\alpha'\leqslant\alpha}\sum_{k\in\mathbb{N}^d_{>\sigma}}\lambda^{\alpha-\alpha'-|k|_\mathfrak{s}}2^{-i(|k|_s+\alpha')}\end{aligned}$$

Since $|k|_\mathfrak{s} + \alpha' > |k|_\mathfrak{s} - \sigma > 0$ a summation of this bound over $i > j'$ then yields the desired bound on the second term in (2.42):

$$C_\Pi(1 + \|\Gamma\|_\gamma) \sum_{k \in \mathbb{N}^d_{>\sigma}} \sum_{\alpha' \in A \colon \alpha' \leqslant \alpha} \lambda^{\alpha - \alpha' - |k|_\mathfrak{s}} \sum_{i > j'} 2^{-i(|k|_\mathfrak{s} + \alpha')} \lesssim C_\Pi(1 + \|\Gamma\|_\gamma) \cdot \lambda^\alpha .$$

□

A classical example of a regularity structure equipped with a model is the *polynomial regularity structure* $\overline{\mathscr{T}}$ which we define via

$$\overline{\mathcal{T}} := \{X^k \mid k \in \mathbb{N}^d\} \tag{2.43}$$

where we identify $X^0 = \mathbf{1}$. We assign to the symbols X^k the homogeneities $|X^k| = |k|_s$ and define $\overline{A} = |\mathbb{N}^d|_\mathfrak{s}$, so that

$$\overline{\mathcal{T}} = \bigoplus_{\alpha \in \overline{A}} \overline{\mathcal{T}}_\alpha := \bigoplus_{\alpha \in \overline{A}} \operatorname{span}\left\{X^k \mid |k|_\mathfrak{s} = \alpha\right\} ,$$

where span $\{\ldots\}$ denotes the vector space generated by the set in the braces. We define a group $\overline{G} = \{\overline{\Gamma}_h \mid h \in \mathbb{R}^d\}$ with group law $\overline{\Gamma}_h \overline{\Gamma}_{h'} := \overline{\Gamma}_{h+h'}$ for h, $h' \in \mathbb{R}^d$. The action of $\overline{G}$ on $\overline{\mathcal{T}}$ is then fixed by requiring $\overline{\Gamma}_h X^k := (X + h\mathbf{1})^k$ (with the obvious meaning of multiplication on the right hand side).

We can realize a model on $\overline{\mathscr{T}} = (\overline{A}, \overline{\mathcal{T}}, \overline{G})$ via

$$\overline{\Pi}_x X^k(y) = (y - x)^k,\ \overline{\Gamma}_{yx} := \overline{\Gamma}_{y-x} . \tag{2.44}$$

for $x, y \in \mathbb{R}^d$ and $k \in \mathbb{N}^d$.

We will sometimes, after mentioning, make the following assumption, which indicates that the polynomial regularity structure has a special significance within the theory.

Assumption 2.3.12. *[Hai14, Assumption 5.3]*

The regularity structure $\mathscr{T} = (A, \mathcal{T}, G)$ with model (Π, Γ) under consideration contains the polynomial regularity structure $\overline{\mathscr{T}} = (\overline{A}, \overline{\mathcal{T}}, \overline{G})$ in the sense of Definition 2.3.4. The restriction of the model (Π, Γ) to $\overline{\mathcal{T}}$ coincides with $(\overline{\Pi}, \overline{\Gamma})$ as given by (2.44).

We further assume that for every $\alpha \in |\mathbb{N}^d|_\mathfrak{s} \cap A$ we have $\mathcal{T}_\alpha = \overline{\mathcal{T}}_\alpha$.

Given a subsector $\mathcal{V} \subseteq \mathcal{T}$ we write $\mathcal{V} \backslash \overline{\mathcal{T}}$ for the complement of the polynomial regularity structure within $\mathcal{V}$

$$\mathcal{V} \backslash \overline{\mathcal{T}} := \bigoplus_{\alpha \in A \backslash |\mathbb{N}^d|_\mathfrak{s}} \mathcal{V}_\alpha . \tag{2.45}$$

Remark 2.3.13. *In the spirit of Definition 2.3.7 it would actually be more accurate to use instead of* $\mathcal{V}\backslash\overline{\mathcal{T}}$ *the symbol* $\mathcal{V}\backslash\overline{\mathcal{V}}$ *with* $\overline{\mathcal{V}} := \bigoplus_{\alpha\in A\cap|\mathbb{N}^d|_{\mathfrak{s}}} \mathcal{V}_\alpha \subseteq \mathcal{V}$.

The polynomial structure $\overline{\mathscr{T}}$ is the typical example one should have in mind in the theory of regularity structure when it comes to comparison with results from "more classical" analysis. From this perspective the spaces $\mathcal{D}^\gamma$ which we are going to define now (and take from [Hai14, Definition 3.1]) are a generalization of classical Hölder spaces. We will allow for a slightly more general framework for these spaces as in [Hai14]: Let $\mathscr{T} = (A, T, G)$ be a regularity structure with a model (Π, Γ). Suppose $\mathcal{V} \subseteq \mathcal{T}$ is a sector of $\mathscr{T}$ and that $\mathcal{V}\backslash\mathcal{W} \subseteq \mathcal{V}$ is a complement of a sector $\mathcal{W}$ within $\mathcal{V}$ in the sense of Definition 2.3.7, so that in total

$$\mathcal{V}\backslash\mathcal{W} \oplus \mathcal{W} = \mathcal{V} \subseteq \mathcal{T}\,.$$

We then define the space of *modelled distributions with bounds in* $\mathcal{V}\backslash\mathcal{W}$ as follows.

Definition 2.3.14. *Let* $\mathcal{V}$ *and* $\mathcal{V}\backslash\mathcal{W}$ *be as above and let* $\Omega \subseteq \mathbb{R}^d$ *and* $\gamma \in \mathbb{R}$. *We say that a mapping* $F : \Omega \to \mathcal{V}_\gamma^-$ *belongs to* $\mathcal{D}^\gamma(\Omega; \mathcal{V}\backslash\mathcal{W}, \Gamma) = \mathcal{D}^\gamma(\Omega; \mathcal{V}\backslash\mathcal{W})$ *if*

$$\|F\|_{\mathcal{D}^\gamma(\Omega;\mathcal{V}\backslash\mathcal{W})} := \sup_{\alpha\in A_{\mathcal{V}\backslash\mathcal{W}},\, x\in\Omega} \|F_x^\alpha\|_{\mathcal{T}_\alpha} + \sup_{\alpha\in A_{\mathcal{V}\backslash\mathcal{W}},\, x,y\in\Omega,\, x\neq y} \frac{\|F_y^\alpha - \Gamma_{yx}^\alpha F_x\|_{\mathcal{T}_\alpha}}{\|y-x\|_{\mathfrak{s}}^{\gamma-\alpha}} < \infty\,. \tag{2.46}$$

Given two different models (Π, Γ), $(\hat{\Pi}, \hat{\Gamma})$ *and* $F \in \mathcal{D}^\gamma(\Omega; \mathcal{V}\backslash\mathcal{W}, \Gamma)$, $\hat{F} \in \mathcal{D}^\gamma(\Omega; \mathcal{V}\backslash\mathcal{W}, \hat{\Gamma})$ *we also define a "distance"*

$$\begin{aligned} \|F; \hat{F}\|_{\mathcal{D}^\gamma(\Omega;\mathcal{V}\backslash\mathcal{W},\Gamma,\hat{\Gamma})} := & \sup_{\alpha\in A_{\mathcal{V}\backslash\mathcal{W}},\, x\in\Omega} \|F_x^\alpha - \hat{F}_x^\alpha\|_{\mathcal{T}_\alpha} \\ & + \sup_{\alpha\in A_{\mathcal{V}\backslash\mathcal{W}},\, x,y\in\Omega,\, x\neq y} \frac{\|F_y^\alpha - \Gamma_{yx}^\alpha F_x - (\hat{F}_y^\alpha - \hat{\Gamma}_{yx}^\alpha \hat{F}_x)\|_{\mathcal{T}_\alpha}}{\|y-x\|_{\mathfrak{s}}^{\gamma-\alpha}}\,. \end{aligned} \tag{2.47}$$

If $\mathcal{W} = \{0\}$ *we simply write* $\mathcal{D}^\gamma(\Omega; \mathcal{V}) = \mathcal{D}^\gamma(\Omega; \mathcal{V}\backslash\{0\})$, *a similar remark applies for the distance* (2.47).

Remark 2.3.15. ⚠ *Note that the notation "*$\mathcal{V}\backslash\mathcal{W}$*" in* $F \in \mathcal{D}^\gamma(\Omega; \mathcal{V}\backslash\mathcal{W})$ *is supposed to indicate two things, first that* F *takes values in the sector* $\mathcal{V}$ *(and* ***not only*** *in* $\mathcal{V}\backslash\mathcal{W}$*) and moreover that the* semi-norm (2.46) *is finite for its components in* $\mathcal{V}\backslash\mathcal{W}$. *In particular we have*

$$\mathcal{D}^\gamma(\Omega; \mathcal{V}) \subseteq \mathcal{D}^\gamma(\Omega; \mathcal{V}\backslash\mathcal{W})\,.$$

Remark 2.3.16. *Due to the bound* (2.38) *it is enough to take in the second term in* (2.46)/ (2.47) *only pairs* $x, y \in \Omega$ *with* $\|x-y\|_{\mathfrak{s}} \leqslant 1$ *if the first term in* (2.46)/(2.47) *is bounded.*

Remark 2.3.17. *Given a mapping $F : \Omega \to \mathcal{V}$ we will also write $F \in \mathcal{D}^\gamma(\Omega; \mathcal{V}\backslash\mathcal{W})$ if*

$$F^{<\gamma} := \sum_{\alpha\in A:\, \alpha<\gamma} F^\alpha \in \mathcal{D}^\gamma(\Omega; \mathcal{V}\backslash\mathcal{W}) .$$

Remark 2.3.18. *Note that a map $F \in \mathcal{D}^\gamma(\mathbb{R}^d; \mathcal{V}\backslash\mathcal{W})$ satisfies global bounds, which is in contrast to [Hai14] where this notation rather means*

$$F \in \bigcap_{n\geqslant 0} \mathcal{D}^\gamma([-n, n]^d; \mathcal{V}\backslash\mathcal{W} .$$

In a framework that is largely based on Fourier analysis such as the paracontrolled approach it seems natural to assume global bounds first and define local spaces afterwards.

Note that the framework above contains the case of any sector $\mathcal{V}$ (by taking $\mathcal{W} = \{0\}$) and in particular $\mathcal{T}$ (by then taking $\mathcal{V} = \mathcal{T}$). Definition 2.3.14 is meaningful since, by definition of a sector, the (semi-)norm (2.46) only sees entries of F in $\mathcal{V}\backslash\mathcal{W}$. In particular, if $F, \tilde{F} \in \mathcal{D}^\gamma(\Omega; \mathcal{V}\backslash\mathcal{W})$ satisfy $F^\alpha = \tilde{F}^\alpha$ for $\alpha \in A_{\mathcal{V}\backslash\mathcal{W}}$ one has $\|F - \tilde{F}\|_{\mathcal{D}^\gamma(\Omega;\mathcal{V}\backslash\mathcal{W})} = 0$. Our motivation for defining modelled distributions with bounds in $\mathcal{V}\backslash\mathcal{W}$ is that for a regularity structure that satisfies Assumption 2.3.12 we sometimes want to "ignore" the polynomial entries of a modelled distributions. In Chapter 5 below we introduce operators (paraproducts) on modelled distributions $F : \mathbb{R}^d \to \mathcal{V}$ that do not depend on the polynomials entries of F, so that it seems quite natural to only require bounds for F^α with $\alpha \in A_{\mathcal{V}\backslash\bar{\mathcal{T}}}$.

The definition of a modelled distribution $F \in \mathcal{D}^\gamma(\Omega; \mathcal{V}\backslash\mathcal{W})$ implies continuity of every component $F^\alpha \in C(\Omega; \mathcal{V}_\alpha)$ with $\alpha \in A_{\mathcal{V}\backslash\mathcal{W}}$ and the bound $\sup_{\alpha\in A_{\mathcal{V}\backslash\mathcal{W}},\, x\in\mathbb{R}^d} \|F^\alpha(x)\|_{\mathcal{T}_\alpha} < \infty$. We will denote functions $F : \Omega \to \mathcal{V}$ that satisfy these two properties by $C_b(\Omega; \mathcal{V}\backslash\mathcal{W})$ and set

$$\|F\|_{C_b(\Omega;\mathcal{V}\backslash\mathcal{W})} := \sup_{\alpha\in A_{\mathcal{V}\backslash\mathcal{W}},\, x\in\Omega} \|F^\alpha_x\|_{\mathcal{T}_\alpha} ,$$

so that in particular $\mathcal{D}^\gamma(\Omega; \mathcal{V}\backslash\mathcal{W}) \subseteq C_b(\Omega; \mathcal{V}\backslash\mathcal{W})$.

We will make use of the following famous result from [Hai14], known as the *reconstruction theorem.*

Theorem 2.3.19 (Theorem 3.10 in [Hai14])**.** *Consider a regularity structure $\mathscr{T} = (A, \mathcal{T}, G)$ together with a model (Π, Γ) (with local bounds) and let $\Omega \subseteq \mathbb{R}^d$ be an open set. For $\gamma > 0$ there is a unique continuous linear map $\mathcal{R} : \mathcal{D}^\gamma(\Omega; \mathcal{T}) \to \mathcal{C}^{\min A}(\Omega; \mathbb{R})$ (where $\mathcal{C}^{\min A}(\Omega; \mathbb{R})$ is defined in Remark 2.1.25 below) with the property that for $x \in \Omega$*

$$|(\mathcal{R}F - \Pi_x F_x)(\varphi^\lambda_{\cdot-x})| \lesssim \lambda^\gamma \|\Pi\|_{\gamma,\Omega}\, \|F\|_{\mathcal{D}^\gamma(\Omega;\mathcal{T})} \tag{2.48}$$

uniformly over $\varphi \in C_c^\infty(B_{\mathfrak{s}}(0,1))$ *with* $\|\varphi\|_{C^r(\mathbb{R}^d;\mathbb{R})} \leqslant 1$ *and* $\operatorname{supp}\varphi^{\lambda/2}_{\cdot-x} \subseteq \Omega$ *where* $r \in \mathbb{N}$ *is the smallest integer with* $r > -\min A$. *Further given a second model* $(\hat{\Pi}, \hat{\Gamma})$ *on* $\mathscr{T}$ *and some* $\hat{F} \in \mathcal{D}^\gamma(\Omega; \mathcal{T}, \hat{\Gamma})$ *we also have*

$$\begin{aligned}&\Big|\Big(\mathcal{R}F - \Pi_x F_x - (\hat{\mathcal{R}}F - \hat{\Pi}_x F_x)\Big)(\varphi^\lambda_{\cdot-x})\Big| \\ &\lesssim \lambda^\gamma \big(\|\hat{\Pi}\|_{\gamma,\Omega} \|F; \hat{F}\|_{\mathcal{D}^\gamma(\Omega;\mathcal{T},\Gamma,\hat{\Gamma})} + \|F\|_{\mathcal{D}^\gamma(\Omega;\mathcal{T},\Gamma)} \|\Pi - \hat{\Pi}\|_{\gamma,\Omega}\big)\,. \end{aligned} \tag{2.49}$$

Remark 2.3.20. *The reconstruction theorem stated in [Hai14, Theorem 3.10] is for* $\Omega = \mathbb{R}^d$ *(where the condition* $\varphi^{\lambda/2}_{\cdot-x} \subseteq \Omega$ *is superfluous). For the version stated here compare [Hai14, Lemma 6.7] for* (2.48). *For* (2.49) *the argument in the proof of [Hai14, Lemma 6.7] still applies to the derivation of (3.4) in the proof of [Hai14, Theorem 3.10].*

Chapter 3

A toolbox for discrete paracontrolled distributions

In this chapter we show how the paracontrolled approach to solve singular SPDE introduced in [GIP15] and [GP15b] can be adapted to a discrete setup. In particular we show how to pass in this framework to the continuous analogues. In Section 3.1 we first recall some fundamental results about the discrete Fourier transform on Bravais lattices and then consider the more general ultra-distribution set-up we introduced in Chapter 2. Thereafter, we define weighted Besov spaces on Bravais lattices and finally show how to extend discrete functions in these spaces to $\mathbb{R}^d$ via some extension operator. For simplicity we only take isotropic scaling

$$\mathfrak{s} = (1, \ldots, 1)\,,$$

but without a doubt all of these results could be translated to the general, anisotropic case. In Section 3.2 we bring the notion of paraproducts from Section 2.1 into the discrete world and prove corresponding estimates. In Section 3.3 we study the interplay of the generator of a random walk on Bravais lattices with our discrete notion of Besov spaces. Finally, in Section 3.4, we provide tools for discrete Wick calculus on Bravais lattices as a tool to renormalize discrete approximations to singular SPDEs. Most of the content of this chapter is taken from [MP17].

3.1 Littlewood-Paley theory on Bravais lattices

3.1.1 Fourier transform on Bravais lattices

A *Bravais-lattice* in d dimensions consists of the integer combinations of d linearly independent vectors $a_1, \ldots, a_d \in \mathbb{R}^d$, that is

$$\mathcal{G} := \mathbb{Z}\, a_1 + \ldots + \mathbb{Z}\, a_d\,. \tag{3.1}$$

Given a Bravais lattice we define the basis $\widehat{a}_1, \dots, \widehat{a}_d$ of the reciprocal lattice by the requirement

$$\widehat{a}_i \bullet a_j = \delta_{ij}\,, \tag{3.2}$$

and we set $\mathscr{R} := \mathbb{Z}\,\widehat{a}_1 + \dots + \mathbb{Z}\,\widehat{a}_d$. However, we will mostly work with the (centered) parapellelotope which is spanned by the basis vectors $\widehat{a}_1, \dots, \widehat{a}_d$:

$$\begin{aligned}\widehat{\mathcal{G}} &:= [0,1)\,\widehat{a}_1 + \dots + [0,1)\,\widehat{a}_d - \frac{1}{2}(\widehat{a}_1 + \dots + \widehat{a}_d) \\ &= [-1/2, 1/2)\,\widehat{a}_1 + \dots + [-1/2, 1/2)\,\widehat{a}_d\,.\end{aligned}$$

We call $\widehat{\mathcal{G}}$ the *bandwidth* or *Fourier-cell* of $\mathcal{G}$ to indicate that the Fourier transform of a map on $\mathcal{G}$ lives on $\widehat{\mathcal{G}}$, as we will see below. We also identify $\widehat{\mathcal{G}} \simeq \mathbb{R}^d/\mathscr{R}$ and turn $\widehat{\mathcal{G}}$ into an additive group which is invariant under translations by elements in $\mathscr{R}$.

Example 3.1.1. *If we choose the canonical basis vectors $a_1 = e_1, \dots, a_d = e_d$, we have simply*

$$\mathcal{G} = \mathbb{Z}^d\,, \qquad \mathscr{R} = \mathbb{Z}^d\,, \qquad \widehat{\mathcal{G}} = \mathbb{T}^d = [-1/2, 1/2)^d\,.$$

Compare also the left lattice in Figure 3.1.1.

In Figure 3.1.1 we sketched some Bravais lattices $\mathcal{G}$ together with their Fourier cells $\widehat{\mathcal{G}}$. Note that the dashed lines between the points of the lattice are at this point a purely artistic supplement. However, they will become meaningful later on: If we imagine a particle performing a random walk on the lattice $\mathcal{G}$, then the dashed lines could be interpreted as the jumps it is allowed to undertake. From this point of view the lines will be drawn by the diffusion operators we introduce in Section 3.3.

Definition 3.1.2. *Given a Bravais lattice $\mathcal{G}$ as defined in* (3.1) *we write*

$$\mathcal{G}^\varepsilon := \varepsilon \mathcal{G}$$

for the sequence of Bravais lattice we obtain by dyadic rescaling with $\varepsilon = 2^{-N}$, $N \geqslant 0$. Whenever we say a statement (or an estimate) holds for $\mathcal{G}^\varepsilon$ we mean that it holds (uniformly) for all $\varepsilon = 2^{-N}$, $N \geqslant 0$.

Remark 3.1.3. *The restriction to dyadic lattices fits well with the use of Littlewood-Paley theory which is traditionally build from dyadic decomposition. However, it turns out that we do not lose much generality by this. Indeed, all the estimates below will hold uniformly as soon as we know that the scale of our lattice is contained in some interval $(c_1, c_2) \subset\subset (0, \infty)$. Therefore it is sufficient to group the members of any positive null-sequence $(\varepsilon_n)_{n \geqslant 0}$ in dyadic intervals $[2^{-(N+1)}, 2^{-N})$ to deduce the general statement.*

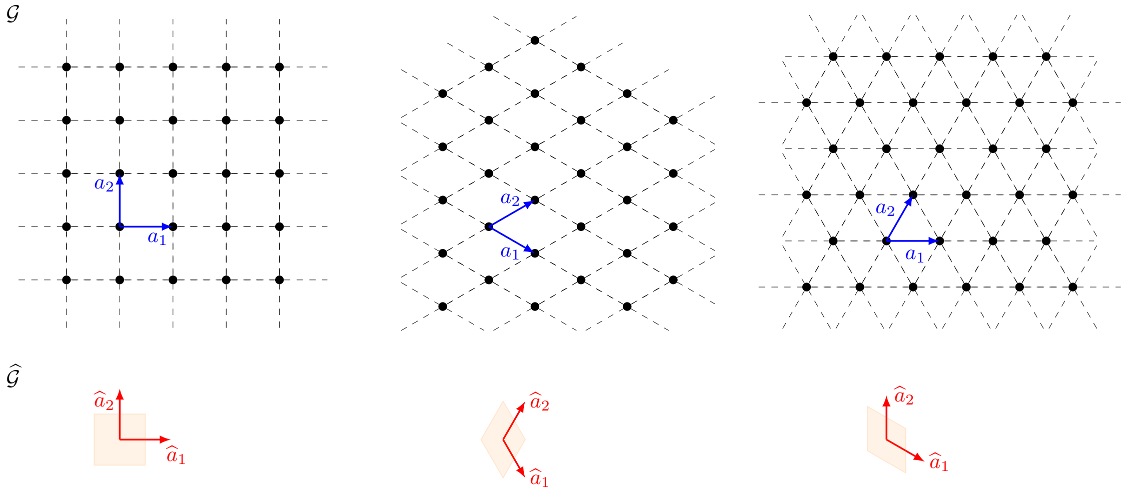

Figure 3.1: Depiction of some Bravais lattices $\mathcal{G}$ with their bandwiths $\widehat{\mathcal{G}}$: a square lattice, an oblique lattice and the so called hexagonal lattice. The length of the reciprocal vectors $\widehat{a}_i$ is rather arbitrary since it actually depends on the units in which we measure a_i.

Given $\varphi \in \ell^1(\mathcal{G})$ we define its Fourier transform as

$$\mathcal{F}_{\mathcal{G}}\varphi(x) := \widehat{\varphi}(x) := |\mathcal{G}| \sum_{k\in\mathcal{G}} \varphi(k) e^{-2\pi\imath k\cdot x}, \qquad x \in \widehat{\mathcal{G}}, \tag{3.3}$$

where we introduced a "normalization constant" $|\mathcal{G}| := |\det(a_1, \ldots, a_d)|$ that ensures that we obtain the usual Fourier transform on $\mathbb{R}^d$ as $|\mathcal{G}|$ tends to 0. For the Fourier cell $\widehat{\mathcal{G}}$ we will write $|\widehat{\mathcal{G}}|$ for the Lebesgue measure of the set $\widehat{\mathcal{G}}$.

If we consider $\mathcal{F}_{\mathcal{G}}\varphi$ as a map on $\mathbb{R}^d$, then it is periodic under translations in $\mathscr{R}$. By the dominated convergence theorem $\mathcal{F}_{\mathcal{G}}\varphi$ is continuous, so since $\widehat{\mathcal{G}}$ is compact it is in $L^1(\widehat{\mathcal{G}}) := L^1(\widehat{\mathcal{G}}, \mathrm{d}x)$, where $\mathrm{d}x$ denotes integration with respect to the Lebesgue measure. For any $\psi \in L^1(\widehat{\mathcal{G}})$ we define its inverse Fourier transform as

$$\mathcal{F}_{\mathcal{G}}^{-1}\psi(k) := \check{\psi}(k) := \int_{\widehat{\mathcal{G}}} \psi(x) e^{2\pi\imath k\cdot x}\mathrm{d}x, \qquad k \in \mathcal{G}. \tag{3.4}$$

Note that $|\mathcal{G}| = 1/|\widehat{\mathcal{G}}|$ and therefore we get at least for φ with finite support $\mathcal{F}_{\mathcal{G}}^{-1}\mathcal{F}_{\mathcal{G}}\varphi = \varphi$. The Schwartz functions on $\mathcal{G}$ are

$$\mathcal{S}(\mathcal{G}) := \left\{\varphi\colon \mathcal{G} \to \mathbb{C} : \sup_{k\in\mathcal{G}}(1+|k|)^m|\varphi(k)| < \infty \text{ for all } m \in \mathbb{N}\right\},$$

and we have $\mathcal{F}_\mathcal{G}\varphi \in C^\infty(\widehat{\mathcal{G}})$ (with periodic boundary conditions) for all $\varphi \in \mathcal{S}(\mathcal{G})$, because for any multi-index $\alpha \in \mathbb{N}^d$ the dominated convergence theorem gives

$$\partial^\alpha \mathcal{F}_\mathcal{G}\varphi(x) = |\mathcal{G}| \sum_{k\in\mathcal{G}} \varphi(k)(-2\pi \imath k)^\alpha e^{-2\pi\imath k\cdot x}.$$

By the same argument we have $\mathcal{F}_\mathcal{G}^{-1}\psi \in \mathcal{S}(\mathcal{G})$ for all $\psi \in C^\infty(\widehat{\mathcal{G}})$, and as in the classical case $\mathcal{G} = \mathbb{Z}^d$ one can show that $\mathcal{F}_\mathcal{G}$ is an isomorphism from $\mathcal{S}(\mathcal{G})$ to $C^\infty(\widehat{\mathcal{G}})$ with inverse $\mathcal{F}_\mathcal{G}^{-1}$. Many relations known from the $\mathbb{Z}^d$-case carry over readily to Bravais lattices such as Parseval's identity

$$\sum_{k\in\mathcal{G}} |\mathcal{G}| \cdot |\varphi(k)|^2 = \int_{|\widehat{\mathcal{G}}|} |\widehat{\varphi}(x)|^2 \,\mathrm{d}x\,. \tag{3.5}$$

(to see this check for example with the Stone-Weierstrass theorem that $(|\mathcal{G}|^{1/2}e^{2\pi\imath k\cdot})_{k\in\mathcal{G}}$ forms an orthonormal basis of $L^2(\widehat{\mathcal{G}}, \mathrm{d}x)$) and the relation between convolution and multiplication

$$\mathcal{F}_\mathcal{G}\left(\varphi_1 *_\mathcal{G} \varphi_2\right)(x) = \mathcal{F}_\mathcal{G}\left(\sum_{k\in\mathcal{G}} |\mathcal{G}|\, \varphi_1(k)\varphi_2(\cdot - k)\right)(x) = \mathcal{F}_\mathcal{G}\varphi_1(x)\cdot \mathcal{F}_\mathcal{G}\varphi_2(x), \tag{3.6}$$

$$\mathcal{F}_\mathcal{G}^{-1}\left(\psi_2 *_{\widehat{\mathcal{G}}} \psi_2\right)(k) := \mathcal{F}_\mathcal{G}^{-1}\left(\int_{\widehat{\mathcal{G}}} \psi_1(x)\psi_2(\cdot - x)\mathrm{d}x\right)(k) = \mathcal{F}_\mathcal{G}^{-1}\psi_1(k)\cdot \mathcal{F}_\mathcal{G}^{-1}\psi_2(k). \tag{3.7}$$

Since $\mathcal{S}(\mathcal{G})$ consists of functions decaying faster than any polynomial, the Schwartz distributions on $\mathcal{G}$ are the functions that grow at most polynomially,

$$\mathcal{S}'(\mathcal{G}) := \left\{ f\colon \mathcal{G} \to \mathbb{C} : \sup_{k\in\mathcal{G}}(1 + |k|)^{-m}|f(k)| < \infty \text{ for some } m \in \mathbb{N}\right\},$$

and $f(\varphi) := |\mathcal{G}| \sum_{k\in\mathcal{G}} f(k)\varphi(k)$ is well defined for $\varphi \in \mathcal{S}(\mathcal{G})$. We extend the Fourier transform to $\mathcal{S}'(\mathcal{G})$ by setting

$$(\mathcal{F}_\mathcal{G} f)(\psi) := \widehat{f}(\psi) := f\left(\overline{\mathcal{F}_\mathcal{G}^{-1}\overline{\psi}}\right) = |\mathcal{G}| \sum_{k\in\mathcal{G}} f(k)\overline{\mathcal{F}_\mathcal{G}^{-1}\overline{\psi}}(k), \qquad \psi \in C^\infty(\widehat{\mathcal{G}}),$$

where $\overline{\cdot}$ denotes the complex conjugate. This should be read as $(\mathcal{F}_\mathcal{G} f)(\psi) = f(\mathcal{F}_\mathcal{G}\psi)$, which however does not make any sense because for $\psi \in C^\infty(\widehat{\mathcal{G}})$ we did not define the Fourier transform $\mathcal{F}_\mathcal{G}\psi$ but only $\mathcal{F}_\mathcal{G}^{-1}\psi$. The Fourier transform $(\mathcal{F}_\mathcal{G} f)(\psi)$ agrees with $\int_{\widehat{\mathcal{G}}} \widehat{f}(x)\psi(x)\mathrm{d}x$ in case $f \in \mathcal{S}(\mathcal{G})$. It is possible to show that $\widehat{f} \in \mathcal{S}'(\widehat{\mathcal{G}})$, where

$$\mathcal{S}'(\widehat{\mathcal{G}}) :=\{u\colon C^\infty(\widehat{\mathcal{G}}) \to \mathbb{C} : u \text{ is linear and } \exists\, C > 0, m \in \mathbb{N}_0 \text{ s.t. } |u(\psi)| \leqslant C\|\psi\|_{C_b^m(\widehat{\mathcal{G}})}\}$$

for $\|\psi\|_{C_b^m(\widehat{\mathcal{G}})} := \sum_{|\alpha|\leqslant m} \|\partial^\alpha \psi\|_{L^\infty(\widehat{\mathcal{G}})}$, and that $\mathcal{F}_{\mathcal{G}}$ is an isomorphism from $\mathcal{S}'(\mathcal{G})$ to $\mathcal{S}'(\widehat{\mathcal{G}})$ with inverse

$$(\mathcal{F}_{\mathcal{G}}^{-1} u)(\varphi) := (\check{u})(\varphi) := |\mathcal{G}| \sum_{k\in\mathcal{G}} u(e^{2\pi\imath k\cdot(\cdot)})\varphi(k). \tag{3.8}$$

As in the classical case $\mathcal{G} = \mathbb{Z}$ it is easy to see that we can identify every $f \in \mathcal{S}'(\mathcal{G})$ with a "dirac comb" distribution $f_{\mathrm{dir}} \in \mathcal{S}'(\mathbb{R}^d)$ by setting

$$f_{\mathrm{dir}} = |\mathcal{G}| \sum_{k\in\mathcal{G}} f(k)\delta(\cdot - k),$$

where $\delta(\cdot - k) \in \mathcal{S}'(\mathbb{R}^d)$ denotes a shifted Dirac delta distribution. We can identify any element $g \in \mathcal{S}'(\widehat{\mathcal{G}})$ of the frequency space with an $\mathscr{R}$-periodic distribution $g_{\mathrm{ext}} \in \mathcal{S}'(\mathbb{R}^d)$ by setting

$$g_{\mathrm{ext}}(\varphi) := g\left(\sum_{k\in\mathscr{R}} \varphi(\cdot - k)\right), \qquad \varphi \in \mathcal{S}(\mathbb{R}^d)\,. \tag{3.9}$$

If $g \in \mathcal{S}'(\widehat{\mathcal{G}})$ coincides with a periodic function on $\widehat{\mathcal{G}}$ one sees that

$$g_{\mathrm{ext}}(x) = g([x]_{\widehat{\mathcal{G}}}) \tag{3.10}$$

where $[x]_{\widehat{\mathcal{G}}}$ is the (unique) element $[x]_{\widehat{\mathcal{G}}} \in \widehat{\mathcal{G}}$ such that $[x] - x \in \mathbb{Z}\hat{a}_1 + \ldots + \mathbb{Z}\hat{a}_d = \mathscr{R}$. Conversely, every $\mathscr{R}$-periodic distribution $g \in \mathcal{S}'(\mathbb{R}^d)$ can be seen as a restricted element $g_{\mathrm{res}} \in \mathcal{S}'(\widehat{\mathcal{G}})$, e.g. by considering

$$g_{\mathrm{res}}(\varphi) := (\psi \cdot g)(\varphi_{\mathrm{ext}}) = g(\psi \cdot \varphi_{\mathrm{ext}}), \qquad \varphi \in C^\infty(\widehat{\mathcal{G}}) \tag{3.11}$$

where $\psi \in C_c^\infty(\mathbb{R}^d)$ is chosen such that $\sum_{k\in\mathscr{R}} \psi(\cdot - k) = 1$ and where we used in the second equality the definition of the product between a smooth function and a distribution. The identification g_{res} does not depend on the choice of ψ as can be easily checked and it motivates our definition of the extension operator $\mathscr{E}$ below in Lemma 3.1.5.

With these identifications in mind we can interpret the concepts introduced above as a sub-theory of the well-known Fourier analysis of tempered distributions. We will sometimes use the following identity for $f \in \mathcal{S}'(\mathcal{G})$

$$(\mathcal{F}_{\mathcal{G}} f)_{\mathrm{ext}} = \mathcal{F}_{\mathbb{R}^d}(f_{\mathrm{dir}})\,, \tag{3.12}$$

which is easily checked using the definitions above.

Next, we want to introduce Besov spaces on $\mathcal{G}$. As in Section 2.1 of Chapter 2 we make use of a dyadic partition of unity $(\varphi_j)_{j\geqslant -1}$, where the support of φ_j

is contained in a rectangular annulus $2^j\mathcal{A}$. Our aim is to define Littlewood-Paley blocks $\Delta_j = \varphi_j(D)$ as in (2.19). In our case all the information about some $f \in \mathcal{S}'(\mathcal{G})$ is stored in a finite bandwidth $\widehat{\mathcal{G}}$ and the Fourier transform $\widehat{f}$ is periodic under translations in $\mathcal{R}$. Therefore, it is more natural to decompose only the compact set $\widehat{\mathcal{G}}$, and we could simply consider finitely many blocks $\Delta_j f$. However, there is a small but delicate problem: We should decompose $\widehat{\mathcal{G}}$ in a smooth periodic way, but if j is such that the support of φ_j touches the boundary of $\widehat{\mathcal{G}}$, the function φ_j will not necessarily be smooth in a periodic sense. Given a dyadic partition of unity as on page 29 we define a dyadic partition of unity associated to a Bravais lattice $\mathcal{G}$ for $x \in \widehat{\mathcal{G}}$ as

$$\varphi_j^{\mathcal{G}}(x) = \begin{cases} \varphi_j(x), & j < j_{\mathcal{G}}\,, \\ 1 - \sum_{j<j_{\mathcal{G}}} \varphi_j(x), & j = j_{\mathcal{G}}\,, \end{cases} \tag{3.13}$$

where $j \leqslant j_{\mathcal{G}} := \inf\{j \,:\, \operatorname{supp}\varphi_j \cap \partial\widehat{\mathcal{G}} \neq \varnothing\}$. We assume for convenience that the used partition of unity $(\varphi_j)_{j\geqslant -1}$ is such that $j_{\mathcal{G}} > 0$, which is always possible due to Remark 2.1.13.

Whenever we take a sequence of lattices $\mathcal{G}^\varepsilon$ as in Definition 3.1.2 we construct all associated Littlewood-Paley decompositions $(\varphi_j^{\mathcal{G}^\varepsilon})_{-1,\ldots,j_{\mathcal{G}^\varepsilon}}$ from the same dyadic decomposition $(\varphi_j)_{j\geqslant -1}$ on $\mathbb{R}^d$.

Now we can define a Littlewood-Paley block for $f \in \mathcal{S}'(\mathcal{G})$ as

$$\Delta_j^{\mathcal{G}} f := \mathcal{F}_{\mathcal{G}}^{-1}(\varphi_j^{\mathcal{G}} \cdot \mathcal{F}_{\mathcal{G}} f).$$

so that $f \in \mathcal{S}'(\mathcal{G})$ can be decomposed in finitely many Littlewood-Paley blocks

$$f = \sum_{-1\leqslant j \leqslant j_{\mathcal{G}}} \Delta_j^{\mathcal{G}} f \tag{3.14}$$

As in the continuous case we will also use the notation $S_j^{\mathcal{G}} f = \sum_{i<j} \Delta_i^{\mathcal{G}} f$ for $j \leqslant j_{\mathcal{G}}$.

Definition 3.1.4. *Given a Bravais lattice $\mathcal{G}$ and parameters $\gamma \in \mathbb{R}$ and $p, q \in [1, \infty]$ we define the* discrete Besov space *on $\mathcal{G}$ by*

$$\mathcal{B}_{p,q}^{\gamma}(\mathcal{G}) := \{f \in \mathcal{S}'(\mathcal{G}) \,|\, \|f\|_{\mathcal{B}_{p,q}^{\gamma}(\mathcal{G})} = \|(2^{j\gamma}\|\Delta_j^{\mathcal{G}} f\|_{L^p(\mathcal{G})})_j\|_{\ell^q} < \infty\},$$

where we define the $L^p(\mathcal{G})$ norm by

$$\|f\|_{L^p(\mathcal{G})} := \left(|\mathcal{G}| \sum_{k\in\mathcal{G}} |f(k)|^p\right)^{1/p} = \||\mathcal{G}|^{1/p} f\|_{\ell^p}. \tag{3.15}$$

We write furthermore $\mathcal{C}_p^{\gamma}(\mathcal{G}) := \mathcal{B}_{p,\infty}^{\gamma}(\mathcal{G})$.

The reader may have noticed that since we only consider finitely many $j = -1, \dots, j_{\mathcal{G}}$, the two spaces $\mathcal{B}^{\gamma}_{p,q}(\mathcal{G})$ and $L^p(\mathcal{G})$ are in fact identical with equivalent norms! However, what we are really after are uniform bounds on sequences $\mathcal{G}^{\varepsilon}$ as in Definition 3.1.2, so that we are of course not allowed to switch between these spaces.

With the above constructions at hand it is easy to develop a theory of paracontrolled distributions on $\mathcal{G}$ which is completely analogous to the one on $\mathbb{R}^d$. To prove the convergence of rescaled lattice models to models on the Euclidean space $\mathbb{R}^d$ we need to compare discrete and continuous distributions, so we are in need of some extension operation. One way of doing so is to simply consider for $f \in \mathcal{S}'(\mathcal{G})$ the identification with a Dirac comb, already mentioned above: $f_{\mathrm{dir}} = |\mathcal{G}| \sum_{k \in \mathcal{G}} f(k)\delta(\cdot - k) \in \mathcal{S}'(\mathbb{R}^d)$, but this has the disadvantage that the extension can only be controlled in spaces of quite low regularity because the Dirac delta has low regularity. We find the following extension convenient:

Lemma 3.1.5. *Let $\mathcal{G}$ be a Bravais lattice, $f \in \mathcal{S}'(\mathcal{G})$ and let $\psi \in C^{\infty}_c(\mathbb{R}^d)$ be a positive function with $\sum_{k \in \mathcal{R}} \psi(\cdot - k) \equiv 1$ and set*

$$\mathscr{E} f := \mathcal{F}^{-1}_{\mathbb{R}^d}(\psi \cdot (\mathcal{F}_{\mathcal{G}} f)_{\mathrm{ext}}), \qquad f \in \mathcal{S}'(\mathcal{G}),$$

where $(\cdot)_{\mathrm{ext}}$ is defined as in (3.9). Then $\mathscr{E} f \in C^{\infty}(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$ and $\mathscr{E} f(k) = f(k)$ for all $k \in \mathcal{G}$.

Proof. We have $\mathscr{E} f \in \mathcal{S}'(\mathbb{R}^d)$ because the periodic extension $(\mathcal{F}_{\mathcal{G}} f)_{\mathrm{ext}}$ of $\mathcal{F}_{\mathcal{G}} f$ is in $\mathcal{S}'(\mathbb{R}^d)$, and therefore also $\mathscr{E} f = \mathcal{F}^{-1}_{\mathbb{R}^d}(\psi\, (\mathcal{F}_{\mathcal{G}} f)_{\mathrm{ext}}) \in \mathcal{S}'(\mathbb{R}^d)$. Knowing that $\mathscr{E} f$ is in $\mathcal{S}'(\mathbb{R}^d)$, it must be in $C^{\infty}(\mathbb{R}^d)$ as well because it has compact spectral support by definition. Moreover, we can write for $k \in \mathcal{G}$

$$\begin{aligned} \mathscr{E}(f)(k) &= (\mathcal{F}_{\mathcal{G}} f)_{\mathrm{ext}}\,(\psi e^{2\pi\imath k\bullet(\cdot)}) = \mathcal{F}_{\mathcal{G}} f\left(\sum_{r \in \mathscr{R}} \psi(\cdot - r) e^{2\pi\imath k\bullet(\cdot - r)}\right) \\ &\overset{(*)}{=} \mathcal{F}_{\mathcal{G}} f(e^{2\pi\imath k\bullet(\cdot)}) = f(k)\,, \end{aligned}$$

where we used in $(*)$ that $k\bullet r \in \mathbb{Z}$ for all $k \in \mathcal{G}$, $r \in \mathscr{R}$ and that $\sum_{r \in \mathscr{R}} \psi(\cdot - r) = 1$. □

As we will see below, it is possible to show that if $\mathscr{E}^{\varepsilon}$ denotes the extension operator on $\mathcal{G}^{\varepsilon}$ as in Definition 3.1.2, then the family $(\mathscr{E}^{\varepsilon})_{\varepsilon > 0}$ is uniformly bounded as linear operators from $\mathcal{B}^{\gamma}_{p,q}(\mathcal{G}^{\varepsilon})$ to $\mathcal{B}^{\gamma}_{p,q}(\mathbb{R}^d)$). This can be used to obtain uniform regularity bounds for the extensions of a given family of lattice models.

However, since we are interested in equations with spatially homogeneous noise, we cannot expect the solution to be in $\mathcal{B}^{\gamma}_{p,q}(\mathcal{G})$ for any γ, p, q and instead we have to consider weighted spaces as in Section 2.1. In the case of the parabolic Anderson model it turns out to be convenient to even allow for subexponential growth of the form $e^{|\cdot|^{\sigma}}$ for $\sigma \in (0, 1)$, which means that we have to work in the ultra-distribution framework we introduced in Section 2.1.

Ultra-distributions on Bravais lattices

For a discrete version of ultra-distributions on a Bravais lattice $\mathcal{G}$ we essentially combine the ideas from Subsection 3.1.1 with those from Section 2.1 and define for $\omega \in \boldsymbol{\omega}$

$$\mathcal{S}_\omega(\mathcal{G}) = \left\{ \varphi : \mathcal{G} \to \mathbb{C} \,\middle|\, \sup_{k\in\mathcal{G}} e^{\lambda\omega(k)} |\varphi(k)| < \infty \text{ for all } \lambda > 0 \right\},$$

and its dual (when equipped with the natural topology)

$$\mathcal{S}'_\omega(\mathcal{G}) = \left\{ f : \mathcal{G} \to \mathbb{C} \,\middle|\, \sup_{k\in\mathcal{G}} e^{-\lambda\omega(k)} |f(k)| < \infty \text{ for some } \lambda > 0 \right\},$$

with the pairing $f(\varphi) = |\mathcal{G}| \sum_{k\in\mathcal{G}} f(k)\varphi(k)$, $\varphi \in \mathcal{S}_\omega(\mathcal{G})$. As above we can then define a Fourier transform on $\mathcal{S}'_\omega(\mathcal{G})$ which maps the discrete space $\mathcal{S}_\omega(\mathcal{G})$ into the space of ultra-differentiable functions

$$\mathcal{S}_\omega(\widehat{\mathcal{G}}) := C^\infty_\omega(\widehat{\mathcal{G}})$$

with *periodic boundary conditions*. The dual space $\mathcal{S}'_\omega(\widehat{\mathcal{G}})$ can then be equipped with a Fourier transform $\mathcal{F}^{-1}_\mathcal{G}$ as in (3.8) such that $\mathcal{F}_\mathcal{G}, \mathcal{F}^{-1}_\mathcal{G}$ become isomorphisms between $\mathcal{S}'_\omega(\mathcal{G})$ and $\mathcal{S}'_\omega(\widehat{\mathcal{G}})$ that are inverse to each other. For a proof of these statements we refer to Lemma 3.5.2 below.

Performing identifications as in the case of $\mathcal{S}'(\mathbb{R}^d)$ we can see these concepts as a sub-theory of the Fourier analysis on $\mathcal{S}'_\omega(\mathbb{R}^d)$ introduced in section 2.1 with the only difference that we have to choose the function ψ with $\sum_{k\in\mathscr{R}} \psi(\cdot - k) = 1$ on page 59 as an element of $C^\infty_{\omega,c}(\mathbb{R}^d)$.

3.1.2 Discrete weighted Besov spaces

We can now give our definition of a discrete, weighted Besov space, where we essentially proceed as in Subsection 3.1.1 with the only difference that $\rho \in \boldsymbol{\rho}(\omega)$ is included in the definition and that the partition of unity $(\varphi_j)_{j\geqslant -1}$, from which $(\varphi^\mathcal{G}_j)_{j\geqslant -1}$ is constructed as on page 60, must now be chosen in $C^\infty_{\omega,c}(\mathbb{R}^d)$.

Definition 3.1.6. *Given a Bravais lattice $\mathcal{G}$, parameters $\gamma \in \mathbb{R}$, $p, q \in [1,\infty]$ and a weight $\rho \in \boldsymbol{\rho}(\omega)$ for $\omega \in \boldsymbol{\omega}$ we define*

$$\mathcal{B}^\gamma_{p,q}(\mathcal{G}, \rho) := \{ f \in \mathcal{S}'_\omega(\mathcal{G}) \mid \|f\|_{\mathcal{B}^\gamma_{p,q}(\mathcal{G},\rho)} := \|(2^{j\gamma} \|\rho \Delta^\mathcal{G}_j f\|_{L^p(\mathcal{G})})_j\|_{\ell^q} < \infty \},$$

where the Littlewood-Paley blocks $(\Delta^\mathcal{G}_j)_{-1,\dots,j_\mathcal{G}}$ are build from a dyadic partition of unity $(\varphi^\mathcal{G}_j)_{j=-1,\dots,j_\mathcal{G}} \subseteq C^\infty_\omega(\widehat{\mathcal{G}})$ constructed as explained above. We write furthermore $\mathcal{C}^\gamma_p(\mathcal{G},\rho) = \mathcal{B}^\gamma_{p,\infty}(\mathcal{G},\rho)$ and define

$$L^p(\mathcal{G}, \rho) := \{ f \in \mathcal{S}_\omega(\mathcal{G}) \mid \|f\|_{L^p(\mathcal{G},\rho)} := \|\rho f\|_{L^p(\mathcal{G})} < \infty \},$$

i.e. $\|f\|_{\mathcal{B}^\gamma_{p,q}(\mathcal{G},\rho)} = \|(2^{j\gamma} \|\Delta^\mathcal{G}_j f\|_{L^p(\mathcal{G},\rho)})_j\|_{\ell^q}$.

As in Section 2.1 we can write the Littlewood-Paley blocks as a convolution (on $\mathcal{G}$) with some functions $\Psi^{\mathcal{G},j} := \mathcal{F}_{\mathcal{G}}^{-1}\varphi_j^{\mathcal{G}}$

$$\Delta_j^{\mathcal{G}} f(x) = \Psi^{\mathcal{G},j} *_{\mathcal{G}} f(x) = |\mathcal{G}| \sum_{k\in\mathcal{G}} \Psi^{\mathcal{G},j}(x-k)f(k)\,, \tag{3.16}$$

for $x \in \mathcal{G}$. Due to our convention to only consider dyadic scalings we always have the following usefull property for a lattice sequence $\mathcal{G}^\varepsilon$ as in Definition 3.1.2

$$\Psi^{\mathcal{G}^\varepsilon,j} = 2^{jd}\phi_{\langle j\rangle_\varepsilon}(2^j\cdot) \tag{3.17}$$

where

$$\langle j\rangle_\varepsilon = \begin{cases} -1, & j=-1 \\ 0, & -1<j<j_{\mathcal{G}^\varepsilon} \\ \infty, & j=j_{\mathcal{G}^\varepsilon} \end{cases}. \tag{3.18}$$

and where ϕ_{-1}, ϕ_0, $\phi_\infty \in \mathcal{S}(\mathbb{R}^d)$ are Schwartz functions *on* $\mathbb{R}^d$ with $\mathcal{F}_{\mathbb{R}^d}\phi_{\langle j\rangle_\varepsilon} \in C^\infty_{\omega,c}(\mathbb{R}^d)$. The functions ϕ_{-1}, ϕ_0, ϕ_∞ dependent on the lattice $\mathcal{G}$ used to construct $\mathcal{G}^\varepsilon = \varepsilon\mathcal{G}$ but are independent of ε. In a way, this is a discrete substitute for the scaling of Ψ^j one finds on $\mathbb{R}^d$, compare Lemma 2.1.14. We prove the identity (3.17) in Lemma 3.1.11 below. It turns out that (3.17) is helpful in translating arguments from the continuous theory into our discrete framework. Let us once more stress the fact that $\phi_{\langle j\rangle_\varepsilon}$ is defined on all of $\mathbb{R}^d$, and therefore (3.16) actually makes sense for all $x \in \mathbb{R}^d$. If one chooses $\phi_{\langle j\rangle_\varepsilon} \in \mathcal{S}_\omega(\mathbb{R}^d)$ as in the proof of Lemma 3.1.11 this smooth extension coincides with the extension $\mathscr{E}^\varepsilon(\Delta_j^{\mathcal{G}} f)$, where the extension operator $\mathscr{E}^\varepsilon$ is defined as in Lemma 3.1.10 below.

The following Lemma can be seen as the discrete analogue of Lemma 2.1.10

Lemma 3.1.7. *Given $\mathcal{G}^\varepsilon$ as in Definition 3.1.2 and $\Phi \in \mathcal{S}_\omega(\mathbb{R}^d)$ for some $\omega \in \boldsymbol{\omega}$ we have for any $\delta \in (0,1]$ with $\delta \gtrsim \varepsilon$ and $p \in [1,\infty]$, $\lambda > 0$ for $\Phi^\delta := \delta^{-d}\Phi(\delta^{-1}\cdot)$*

$$\|\Phi^\delta\|_{L^p(\mathcal{G}^\varepsilon, e^{\lambda\omega})} \lesssim \delta^{-d(1-1/p)}\,,$$

where the implicit constant is independent of $\varepsilon > 0$. We even have the stronger result

$$\sup_{x\in\mathbb{R}^d} \|\Phi^\delta(\cdot + x)\|_{L^p(\mathcal{G}^\varepsilon, e^{\lambda\omega(\cdot+x)})} \lesssim \delta^{-d(1-1/p)}\,. \tag{3.19}$$

In particular we have for $\rho \in \boldsymbol{\rho}(\omega)$

$$\|\Phi^\delta *_{\mathcal{G}^\varepsilon} f\|_{L^p(\mathcal{G}^\varepsilon,\rho)} \lesssim \|f\|_{L^p(\mathcal{G}^\varepsilon,\rho)}, \qquad \|\Phi^\delta *_{\mathcal{G}^\varepsilon} f\|_{L^p(\mathbb{R}^d,\rho)} \lesssim \|f\|_{L^p(\mathcal{G}^\varepsilon,\rho)}\,, \tag{3.20}$$

where we used in the second estimate that

$$x \mapsto (\Phi^\delta *_{\mathcal{G}^\varepsilon} f)(x) = |\mathcal{G}^\varepsilon| \sum_{k \in \mathcal{G}^\varepsilon} \Phi^\delta(x-k)\, f(k)$$

can be extended to $\mathbb{R}^d$.

Remark 3.1.8. *Using* $\delta = 2^{-j}$ *for* $j \in \{-1, \ldots, j_{\mathcal{G}^\varepsilon}\}$ *this covers in particular the functions* $\Psi^{\mathcal{G}^\varepsilon, j} = \mathcal{F}^{-1}_{\mathcal{G}^\varepsilon} \varphi_j^{\mathcal{G}^\varepsilon}$ *via* (3.17).

Proof. The case $p = \infty$ follows from the definition of $\mathcal{S}_\omega(\mathbb{R}^d)$ and $e^{\lambda\omega(k)} \leqslant e^{\lambda\omega(\delta^{-1}k)}$, so that we only have to show the statement for $p < \infty$. And indeed we obtain

$$\begin{aligned}
\|\Phi^\delta\|^p_{L^p(\mathcal{G}^\varepsilon, e^{\lambda\omega})} &= \sum_{k \in \mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon| |\Phi^\delta(k)|^p e^{p\lambda\omega(k)} = \delta^{-dp}\varepsilon^d \sum_{k \in \mathcal{G}} |\mathcal{G}||\Phi(\delta^{-1}\varepsilon k)|^p e^{p\lambda\omega(\varepsilon k)} \\
&\leqslant \delta^{-dp}\varepsilon^d \sum_{k \in \mathcal{G}} |\mathcal{G}||\Phi(\delta^{-1}\varepsilon k)|^p e^{p\lambda\omega(\delta^{-1}\varepsilon k)} \\
&\lesssim \delta^{-d(p-1)} \sum_{k \in \mathcal{G}} |\mathcal{G}| \delta^{-d}\varepsilon^d \frac{1}{1 + |\delta^{-1}\varepsilon k|^{d+1}} \\
&\overset{\text{Lemma 3.5.1}}{\lesssim} \delta^{-d(p-1)} \int_{\mathbb{R}^d} \mathrm{d}z\, (\delta^{-1}\varepsilon)^d \frac{1}{1 + |\delta^{-1}\varepsilon z|^{d+1}} \lesssim \delta^{-d(p-1)}\,,
\end{aligned}$$

where we used that $\Phi \in \mathcal{S}_\omega(\mathbb{R}^d)$ and in the application of Lemma 3.5.1 that for $|x - y| \lesssim 1$ the quotient $\frac{1+|\delta^{-1}\varepsilon x|}{1+|\delta^{-1}\varepsilon y|}$ is uniformly bounded. Inequality (3.19) can be proved in the same way since it suffices to take the supremum over $|x| \lesssim \varepsilon$.

The estimates on $\Phi^\delta *_{\mathcal{G}^\varepsilon} f$ then follow by Young's inequality on $\mathcal{G}^\varepsilon$ and a mixed Young inequality, Lemma 3.5.3 below, together with (2.2). □

As in the continuous case we can state an embedding theorem for discrete Besov spaces. Since it can be shown exactly as its continuous cousin Lemma 2.1.26 we will not give its proof here.

Lemma 3.1.9. *Given* $\mathcal{G}^\varepsilon$ *as in Definition 3.1.2 for any* $\gamma_1 \in \mathbb{R}$, $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$ *and weights* ρ_1, ρ_2 *with* $\rho_2 \lesssim \rho_1$ *we have the continuous embedding (with norm of the embedding operator independent of* $\varepsilon \in (0,1]$*)*

$$\mathcal{B}^{\gamma_1}_{p_1,q_1}(\mathcal{G}^\varepsilon, \rho_1) \subseteq \mathcal{B}^{\gamma_2}_{p_2,q_2}(\mathcal{G}^\varepsilon, \rho_2)$$

for $\gamma_2 - \frac{d}{p_2} \leqslant \gamma_1 - \frac{d}{p_1}$.

The extension operator

Given a Bravais lattice $\mathcal{G}$ and a dyadic partition of unity $(\varphi_j)_{j\geqslant -1}$ on $\mathbb{R}^d$ such that $j_{\mathcal{G}}$ as defined on page 60 is strictly greater than 0 we construct a discrete dyadic partition of unity $(\varphi_j^{\mathcal{G}})_{-1,\dots,j_{\mathcal{G}}}$ from $(\varphi_j)_{j\geqslant -1}$ as on page 60.

We choose a symmetric function $\psi \in C^{\infty}_{\omega,c}(\mathbb{R}^d)$ which we refer to as the *smear function* and which satisfies the following properties:

1. $\sum_{k\in\mathscr{R}} \psi(\cdot - k) = 1$,
2. $\psi = 1$ on $\operatorname{supp}\varphi_j$ for $j < j_{\mathcal{G}}$,
3. $\operatorname{supp}\psi \subseteq B(0,R)$ with $R > 0$ small enough such that

$$\left(B(0,R) \cap \operatorname{supp}\,\left(\varphi_j^{\mathcal{G}}\right)_{\mathrm{ext}}\right)\backslash\widehat{\mathcal{G}} \Rightarrow j = j_{\mathcal{G}}\,.$$

The last property looks slightly technical, but actually only states that the support of ψ is small enough such that it only touches the support of the periodically extended $\varphi_j^{\mathcal{G}}$ with $j < j_{\mathcal{G}}$ *inside* $\widehat{\mathcal{G}}$. Using $\operatorname{dist}(\partial\widehat{\mathcal{G}}, \bigcup_{j<j_{\mathcal{G}}} \operatorname{supp}(\varphi_j^{\mathcal{G}})_{\mathrm{ext}}) > 0$ it is not hard to construct a function ψ as above via Lemma 2.1.9.

The rescaled $\psi^{\varepsilon} := \psi(\varepsilon\cdot)$ satisfies the same properties on $\mathcal{G}^{\varepsilon}$ (remember that by convention we construct the sequence $(\varphi_j^{\mathcal{G}^{\varepsilon}})_{j=-1,\dots,j_{\mathcal{G}^{\varepsilon}}}$ from the same $(\varphi_j)_{j\geqslant -1}$). This allows us to define an extension operator $\mathscr{E}^{\varepsilon}$ in the spirit of Lemma 3.1.5 as

$$\mathscr{E}^{\varepsilon} f := \mathcal{F}^{-1}_{\mathbb{R}^d}(\psi^{\varepsilon}\cdot(\mathcal{F}_{\mathcal{G}^{\varepsilon}} f)_{\mathrm{ext}}), \qquad f \in \mathcal{S}'_{\omega}(\mathcal{G}^{\varepsilon}),$$

and as in Lemma 3.1.5 we can show that $\mathscr{E}^{\varepsilon} f \in C^{\infty}_{\omega}(\mathbb{R}^d) \cap \mathcal{S}'_{\omega}(\mathbb{R}^d)$ and $\mathscr{E}^{\varepsilon} f|_{\mathcal{G}^{\varepsilon}} = f$.

Using (3.12) we can give a useful, alternative formulation of $\mathscr{E}^{\varepsilon} f$

$$\begin{aligned} \mathscr{E}^{\varepsilon} f &= \mathcal{F}^{-1}_{\mathbb{R}^d}\psi^{\varepsilon} * \mathcal{F}^{-1}_{\mathbb{R}^d}(\mathcal{F}_{\mathcal{G}^{\varepsilon}} f)_{\mathrm{ext}} = \mathcal{F}^{-1}_{\mathbb{R}^d}\psi^{\varepsilon} * f_{\mathrm{dir}} \\ &= \mathcal{F}^{-1}_{\mathbb{R}^d}\psi^{\varepsilon} *_{\mathcal{G}^{\varepsilon}} f = |\mathcal{G}^{\varepsilon}| \sum_{z\in\mathcal{G}^{\varepsilon}} \mathcal{F}^{-1}_{\mathbb{R}^d}\psi^{\varepsilon}(\cdot - z) f(z)\,, \end{aligned} \tag{3.21}$$

where we read similar as for (3.16) the convolution in the second line as a function on $\mathbb{R}^d$ using that $\mathcal{F}^{-1}_{\mathbb{R}^d}\psi^{\varepsilon} \in \mathcal{S}_{\omega}(\mathbb{R}^d)$ is defined on $\mathbb{R}^d$. By property 3 of ψ we also have for $j < j_{\mathcal{G}^{\varepsilon}}$

$$\Delta_j \mathscr{E}^{\varepsilon} f = \mathscr{E}^{\varepsilon} \Delta_j^{\mathcal{G}^{\varepsilon}} f \tag{3.22}$$

Finally, let us study the interplay of $\mathscr{E}^{\varepsilon}$ with Besov spaces.

Lemma 3.1.10. *For any $\gamma \in \mathbb{R}$, $p, q \in [1,\infty]$ and $\rho \in \boldsymbol{\rho}(\boldsymbol{\omega})$ the family of operators*

$$\mathscr{E}^{\varepsilon} \colon \mathcal{B}^{\gamma}_{p,q}(\mathcal{G}^{\varepsilon},\rho) \longrightarrow \mathcal{B}^{\gamma}_{p,q}(\mathbb{R}^d,\rho)\,,$$

defined above, is uniformly bounded in ε.

Proof. We have to estimate $\Delta_j \mathscr{E}^\varepsilon f$ for $j \geqslant -1$. For $j < j_{\mathcal{G}^\varepsilon}$ we can apply (3.22) and (3.21) together with Lemma 3.1.7 to bound

$$\begin{aligned}\|\Delta_j \mathscr{E}^\varepsilon f\|_{L^p(\mathbb{R}^d,\rho)} &= \|\varepsilon^{-d}(\mathcal{F}_{\mathbb{R}^d}\psi)(\varepsilon\cdot) *_{\mathcal{G}^\varepsilon} \Delta_j^{\mathcal{G}^\varepsilon} f\|_{L^p(\mathbb{R}^d,\rho)} \lesssim \|\Delta_j^{\mathcal{G}^\varepsilon} f\|_{L^p(\mathcal{G}^\varepsilon,\rho)} \\ &\lesssim 2^{-j\gamma}\|f\|_{\mathcal{B}^\gamma_{p,q}(\mathcal{G}^\varepsilon,\rho)}\end{aligned}$$

For $j \geqslant j_{\mathcal{G}^\varepsilon}$ only $j \sim j_{\mathcal{G}^\varepsilon}$ contributes due to the compact support of ψ^ε. By spectral support properties we have

$$\Delta_j \mathscr{E}^\varepsilon f = \Delta_j(\mathscr{E}^\varepsilon \sum_{i\sim j_{\mathcal{G}^\varepsilon}} \Delta_i^{\mathcal{G}^\varepsilon} f)$$

Using from (2.20) that Δ_j maps $L^p(\mathbb{R}^d,\rho)$ itself we thus obtain

$$\|\Delta_j \mathscr{E}^\varepsilon f\|_{L^p(\mathbb{R}^d,\rho)} \lesssim \|\mathscr{E}^\varepsilon \sum_{i\sim j_{\mathcal{G}^\varepsilon}} \Delta_i^{\mathcal{G}^\varepsilon} f\|_{L^p(\mathcal{G}^\varepsilon,\rho)} \lesssim 2^{-j_{\mathcal{G}^\varepsilon}\gamma}\|f\|_{\mathcal{B}^\gamma_{p,q}(\mathcal{G}^\varepsilon,\rho)}$$

where we applied once more (3.21) and Lemma 3.1.7 in the second step. □

Below, we will often be given some functional $F(f_1,\ldots,f_n)$ on discrete Besov functions taking values in a discrete Besov space X (or some space constructed from it) that satisfies a bound of the type

$$\|F(f_1,\ldots,f_n)\|_X \leqslant c(f_1,\ldots,f_n). \tag{3.23}$$

We then say that the estimate (3.23) has the property $(\mathscr{E})$ (on X) if there is a "continuous version" $\overline{F}$ of F and a continuous version $\overline{X}$ of X and a sequence of constants $o_\varepsilon \to 0$ such that

$$\|\mathscr{E}^\varepsilon F(f_1,\ldots,f_n) - \overline{F}(\mathscr{E}^\varepsilon f_1,\ldots,\mathscr{E}^\varepsilon f_n)\|_{\overline{X}} \leqslant o_\varepsilon \cdot c(f_1,\ldots,f_n) \tag{$\mathscr{E}$}$$

In other words we can pull the operator $\mathscr{E}^\varepsilon$ inside F without paying anything in the limit. With the smear function ψ introduced above when can now also give the proof of the announced scaling property (3.17) of the functions $\Psi^{\mathcal{G}^\varepsilon,j}$.

Lemma 3.1.11. *Let $\mathcal{G}^\varepsilon$ be as in Definition 3.1.2 and let $\omega \in \boldsymbol{\omega}$. Let $(\varphi_j^{\mathcal{G}^\varepsilon})_{j=-1,\ldots,j_{\mathcal{G}^\varepsilon}} \subseteq C^\infty_{\omega,c}(\widehat{\mathcal{G}^\varepsilon})$ be a partition of unity of $\widehat{\mathcal{G}^\varepsilon}$ as defined on page 60 and take $\Psi^{\mathcal{G}^\varepsilon,j} = \mathcal{F}^{-1}_{\mathcal{G}^\varepsilon}\varphi_j^{\mathcal{G}^\varepsilon}$ and $\Psi^{\mathcal{G}^\varepsilon,<j} := \sum_{i<j}\Psi^{\mathcal{G}^\varepsilon,i}$. The extensions*

$$\begin{aligned}\tilde{\Psi}^{\varepsilon,j} &:= \mathscr{E}^\varepsilon\Psi^{\mathcal{G}^\varepsilon,j} = \mathcal{F}^{-1}_{\mathbb{R}^d}(\psi^\varepsilon \cdot (\varphi_j^{\mathcal{G}^\varepsilon})_{\mathrm{ext}}) \\ \tilde{\Psi}^{\varepsilon,<j} &:= \mathscr{E}^\varepsilon\Psi^{\mathcal{G}^\varepsilon,<j} = \mathcal{F}^{-1}_{\mathbb{R}^d}\Big(\psi^\varepsilon \cdot \Big(\sum_{i<j}\varphi_i^{\mathcal{G}^\varepsilon}\Big)_{\mathrm{ext}}\Big)\end{aligned}$$

are elements of $\mathcal{S}_\omega(\mathbb{R}^d)$. *Moreover there are* $\check{\phi}_{-1}, \check{\phi}_0, \check{\phi}_\infty, \check{\phi}_\Sigma \in C^\infty_{\omega,c}(\mathbb{R}^d)$, *independent of* ε, *such that for for* $j = -1, \ldots, j_{\mathcal{G}^\varepsilon}$ *and* $j' = 0, \ldots, j_{\mathcal{G}^\varepsilon}$ *with* $\langle j \rangle_\varepsilon$ *as in* (3.18)

$$\psi^\varepsilon \cdot \big(\varphi_j^{\mathcal{G}^\varepsilon}\big)_{\mathrm{ext}} = \check{\phi}_{\langle j \rangle_\varepsilon}(2^{-j}\cdot)\,, \tag{3.24}$$

$$\psi^\varepsilon \cdot \Big(\sum_{i<j'} \varphi_i^{\mathcal{G}^\varepsilon}\Big)_{\mathrm{ext}} = \check{\phi}_\Sigma(2^{-j'}\cdot)\,. \tag{3.25}$$

The functions $\check{\phi}_0$ *and* $\check{\phi}_\infty$ *have moreover support in a rectangular annulus* $\mathcal{A} \subseteq \mathbb{R}^d$.

In particular we have for $j = -1, \ldots, j_{\mathcal{G}^\varepsilon}$ *and* $j' = 0, \ldots, j_{\mathcal{G}^\varepsilon}$.

$$\tilde{\Psi}^{\varepsilon,j} = 2^{jd} \cdot \phi_{\langle j \rangle_\varepsilon}(2^j\cdot)\,, \qquad \tilde{\Psi}^{\varepsilon,<j'} = 2^{j'd} \cdot \phi_\Sigma(2^{j'}\cdot)$$

where $\phi_i := \mathcal{F}^{-1}_{\mathbb{R}^d}\check{\phi}_i$ *for* $i \in \{-1, 0, \infty, \Sigma\}$.

Proof. We only have to show (3.24) and (3.25). For $j < j_{\mathcal{G}^\varepsilon}$ and $0 \leqslant j' \leqslant j_{\mathcal{G}^\varepsilon}$ we use that by construction of $\varphi_j^{\mathcal{G}^\varepsilon}$ out of a dyadic partition of unity $(\varphi_j)_{j \geqslant -1}$ on $\mathbb{R}^d$ we have inside $\widehat{\mathcal{G}^\varepsilon}$

$$\varphi_j^{\mathcal{G}^\varepsilon} = \varphi_j\,, \qquad \sum_{i<j'} \varphi_i^{\mathcal{G}^\varepsilon} = \sum_{i<j'} \varphi_i$$

so that due to property 2 and 3 of the smear function ψ^ε and (2.16) it is enough to take

$$\check{\phi}_\Sigma = \varphi_{-1}$$

and for $j < j_{\mathcal{G}^\varepsilon}$ by the scaling property of φ_j from (2.15)

$$\check{\phi}_{\langle j \rangle_\varepsilon} := \varphi_j(2^j\cdot) \in \{\varphi_{-1}(\cdot/2),\, \varphi_0\}\,.$$

For the construction of ϕ_∞ a bit more work is required. As in (3.10) let us denote by $[x]_{\widehat{\mathcal{G}^\varepsilon}}$ the unique element $[x]_{\widehat{\mathcal{G}^\varepsilon}} \in \widehat{\mathcal{G}}$ for which for $x \in \mathbb{R}^d$ one has $x - [x]_{\mathcal{G}^\varepsilon} \in \mathscr{R}^\varepsilon$. One then easily checks

$$\varepsilon[x]_{\widehat{\mathcal{G}^\varepsilon}} = [\varepsilon x]_{\widehat{\mathcal{G}}}\,. \tag{3.26}$$

Recall that by definition of our lattice sequence $\mathcal{G}^\varepsilon$ we took dyadic scaling $\varepsilon = 2^{-N}$ which implies in particular

$$2^{-j_{\mathcal{G}^\varepsilon}} = \varepsilon \cdot 2^k \tag{3.27}$$

for some fixed $k \in \mathbb{Z}$. Using once more (2.16) and relation (3.27) we can write for $x \in \widehat{\mathcal{G}^\varepsilon}$

$$\varphi^{\mathcal{G}^\varepsilon}_{j_{\mathcal{G}^\varepsilon}}(x) = 1 - \sum_{j<j_{\mathcal{G}^\varepsilon}} \varphi_j(x) = 1 - \varphi_{-1}(2^{-j_{\mathcal{G}^\varepsilon}}x) = \chi(\varepsilon x)$$

for some $\chi \in C_\omega^\infty(\mathbb{R}^d)$. Applying (3.26) we obtain for $x \in \mathbb{R}^d$ that the periodic extension

$$\varphi_{j_{\mathcal{G}^\varepsilon}}^{\mathcal{G}^\varepsilon}([x]_{\widehat{\mathcal{G}^\varepsilon}}) = \chi(\varepsilon[x]_{\widehat{\mathcal{G}^\varepsilon}}) = \chi([\varepsilon x]_{\widehat{\mathcal{G}}})$$

is the ε scaled version of some smooth, $\mathscr{R}$-periodic function $\chi([\cdot]_{\widehat{\mathcal{G}}}) \in C_\omega^\infty(\widehat{\mathcal{G}})$ (to see that the composition with $[\cdot]_{\widehat{\mathcal{G}}}$ does not change the smoothness, note that χ is 1 in an environment around $\partial\widehat{\mathcal{G}}$). Consequently

$$\psi(\varepsilon\cdot)\left(\varphi_{j_{\mathcal{G}^\varepsilon}}^{\mathcal{G}^\varepsilon}\right)_{\mathrm{ext}} = \left(\psi\chi([\cdot]_{\widehat{\mathcal{G}}})\right)(\varepsilon\cdot)$$

so that setting $\check{\phi}_\infty = \left(\psi\chi([\cdot]_{\widehat{\mathcal{G}}})\right)(2^{-k}\cdot)$ with k as in (3.27) finishes the proof. □

3.2 Discrete Paracontrolled Calculus

Using the discrete decomposition

$$f = \sum_{-1 \leqslant j \leqslant j_{\mathcal{G}}} \Delta_j^{\mathcal{G}} f$$

for a discrete $f \in \mathcal{S}'_\omega(\mathcal{G})$ on a Bravais lattice $\mathcal{G}$ the following versions of the operators $\prec, \circ$, introduced in Section 2.1, follow in a natural way.

Definition 3.2.1. *Given a Bravais lattice $\mathcal{G}$ and discrete ultra-distributions f_1, $f_2 \in \mathcal{S}'_\omega(\mathcal{G})$ for $\omega \in \boldsymbol{\omega}$ we define their* discrete paraproduct *by*

$$f_1 \prec^{\mathcal{G}} f_2 := \sum_{0 < j_2 \leqslant j_{\mathcal{G}}} \sum_{-1 \leqslant j_1 < j_2 - 1} \Delta_{j_1}^{\mathcal{G}} f_1 \cdot \Delta_{j_2}^{\mathcal{G}} f_2\,, \tag{3.28}$$

and we also write $f_1 \succ^{\mathcal{G}} f_2 := f_2 \prec^{\mathcal{G}} f_1$. The discrete resonant product *is*

$$f_1 \circ^{\mathcal{G}} f_2 := \sum_{-1 \leqslant j_1, j_2 \leqslant j_{\mathcal{G}} \colon |j_1 - j_2| \leqslant 1} \Delta_{j_1}^{\mathcal{G}} f_1 \cdot \Delta_{j_2}^{\mathcal{G}} f_2\,. \tag{3.29}$$

If there is no risk for confusion we may drop the index $\mathcal{G}$ on $\prec$, $\succ$, and $\circ$.

In contrast to the continuous theory all these operators are well defined without any further restrictions since they only involve finite sums. However in order to obtain uniform estimates on a lattice sequence $\mathcal{G}^\varepsilon$ the same conditions as in Lemma 2.1.34 arise.

Lemma 3.2.2. *Given a sequence $\mathcal{G}^\varepsilon$ as in Definition 3.1.2, $\rho_1, \rho_2 \in \boldsymbol{\rho}(\omega)$ for some $\omega \in \boldsymbol{\omega}$ and p, p_1, p_2, q, q_1, $q_2 \in [1, \infty]$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$ we have the bounds:*

- *For any $\gamma_2 \in \mathbb{R}$*

$$\|f_1 \prec f_2\|_{\mathcal{B}^{\gamma_2}_{p,q_2}(\mathcal{G}^\varepsilon, \rho_1 \cdot \rho_2)} \lesssim \|f_1\|_{L^{p_1}(\mathcal{G}^\varepsilon, \rho_1)} \|f_2\|_{\mathcal{B}^{\gamma_2}_{p_2,q_2}(\mathcal{G}^\varepsilon, \rho_2)} .$$

- *For any $\gamma_1 < 0$, $\gamma_2 \in \mathbb{R}$*

$$\|f_1 \prec f_2\|_{\mathcal{B}^{\gamma_1+\gamma_2}_{p,q}(\mathcal{G}^\varepsilon, \rho_1 \cdot \rho_2)} \lesssim \|f_1\|_{\mathcal{B}^{\gamma_1}_{p_1,q_1}(\mathcal{G}^\varepsilon, \rho_1)} \|f_2\|_{\mathcal{B}^{\gamma_2}_{p_2,q_2}(\mathcal{G}^\varepsilon, \rho_2)} .$$

- *For any $\gamma_1, \gamma_2 \in \mathbb{R}$ with $\gamma_1 + \gamma_2 > 0$*

$$\|f_1 \circ f_2\|_{\mathcal{B}^{\gamma_1+\gamma_2}_{p,q}(\mathcal{G}^\varepsilon, \rho_1 \cdot \rho_2)} \lesssim \|f_1\|_{\mathcal{B}^{\gamma_1}_{p_1,q_1}(\mathcal{G}^\varepsilon, \rho_1)} \|f_2\|_{\mathcal{B}^{\gamma_2}_{p_2,q_2}(\mathcal{G}^\varepsilon, \rho_2)} .$$

All involved constants only depend on $\mathcal{G}$ but are independent of ε. All estimates have the property ($\mathscr{E}$).

Proof. The proof of the estimates follows along the lines of Lemma 2.1.34.

To check the ($\mathscr{E}$)-property we recall that $\mathscr{E}^\varepsilon g = \mathcal{F}^{-1}_{\mathbb{R}^d}(\psi(\varepsilon \cdot) \cdot (\mathcal{F}_{\mathcal{G}^\varepsilon} g)_{\mathrm{ext}})$ with $\psi(\varepsilon \cdot) = 1$ in some ball of order $\varepsilon^{-1} \approx 2^{-j_{\mathcal{G}^\varepsilon}}$ inside $\widehat{\mathcal{G}^\varepsilon}$ and with $(\ldots)_{\mathrm{ext}}$ denoting the periodic extension as in (3.9). We thus have by the spectral support properties of the paraproduct

$$\begin{aligned} &\Delta_i(\mathscr{E}^\varepsilon(f_1 \prec^{\mathcal{G}^\varepsilon} f_2) - \mathscr{E}^\varepsilon f_1 \prec \mathscr{E}^\varepsilon f_2) \\ &= \mathbf{1}_{i \sim j_{\mathcal{G}^\varepsilon}} \left(\Delta_i \mathscr{E}^\varepsilon \left(\sum_{j \sim i} S^{\mathcal{G}^\varepsilon}_{j-1} f_1 \, \Delta^{\mathcal{G}^\varepsilon}_j f_2 \right) - \Delta_i \left(\sum_{j \sim i} S_{j-1} \mathscr{E}^\varepsilon f_1 \, \Delta_j \mathscr{E}^\varepsilon f_2 \right) \right) . \end{aligned}$$

Together with (3.21), Lemma 3.1.11 and Lemma 3.1.7 this gives for the first two estimates the bounds $\mathbf{1}_{i \sim j_{\mathcal{G}^\varepsilon}} 2^{-i\gamma_2} \lesssim 2^{-i(\gamma_2 - \kappa)} \varepsilon^\kappa$ and $\mathbf{1}_{i \sim j_{\mathcal{G}^\varepsilon}} 2^{-i(\gamma_1+\gamma_2)} \lesssim 2^{-i(\gamma_1+\gamma_2-\kappa)} \varepsilon^\kappa$. For the third case we obtain by similar arguments for $\Delta_i \left(\mathscr{E}^\varepsilon(f_1 \circ^{\mathcal{G}^\varepsilon} f_2) - \mathscr{E}^\varepsilon f_1 \circ \mathscr{E}^\varepsilon f_2\right)$ the bound

$$\begin{aligned} \sum_{j:\, i \lesssim j \sim j_{\mathcal{G}^\varepsilon}} 2^{-j(\gamma_1+\gamma_2)} \epsilon^q_j &\lesssim \sum_{j:\, i \lesssim j \sim j_{\mathcal{G}^\varepsilon}} 2^{-j(\gamma_1+\gamma_2-\kappa)} \epsilon^q_j \, \varepsilon^\kappa \\ &= 2^{-i(\gamma_1+\gamma_2-\kappa)} \sum_{j:\, i \lesssim j \sim j_{\mathcal{G}^\varepsilon}} 2^{(i-j)(\gamma_1+\gamma_2-\kappa)} \epsilon^q_j \, \varepsilon^\kappa , \end{aligned}$$

where ϵ^q_j is a sequence such that $\|\epsilon^q_j\|_{\ell^q} \leqslant 1$ and where $\kappa > 0$ is small enough such that $\gamma_1 + \gamma_2 - \kappa > 0$. The result then follows by Young's inequality for sequences. □

From this Lemma one easily derives the same multiplication bounds as in Corollary 2.1.35.

An important observation in [GIP15] was that if the regularity condition $\gamma_1+\gamma_2 > 0$ is not satisfied in (2.1.34) it may yet still be possible to make sense of $f_1 \circ f_2$ as long as f_1 can be written as a paraproduct plus a smoother remainder. The main lemma which makes this possible is an estimate for a certain commutator. We define the discrete version of the commutator as

$$C^{\mathcal{G}}(f_1, f_2, f_3) := (f_1 \prec^{\mathcal{G}} f_2) \circ^{\mathcal{G}} f_3 - f_1 (f_2 \circ^{\mathcal{G}} f_3) .$$

for some Bravais lattice $\mathcal{G}$ and $f_1,\ f_2,\ f_3 \in \mathcal{S}'_\omega(\mathcal{G})$. If there is no risk for confusion we may drop the index $\mathcal{G}$ on C. In the following lemma we describe how $C^{\mathcal{G}}$ interplays with discrete Besov spaces on the sequence $\mathcal{G}^\varepsilon$. Although this could be formulated in a more general setup we restrict ourselves to the Besov space $\mathcal{C}^\gamma_p = \mathcal{B}^\gamma_{p,\infty}$, since this is all we need in this thesis.

Lemma 3.2.3. *([GP15c, Lemma 14]) Given $\rho_1,\ \rho_2,\ \rho_3 \in \boldsymbol{\rho}(\omega)$ for some $\omega \in \boldsymbol{\omega}$, $p \in [1, \infty]$ and $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ with $\gamma_1 + \gamma_2 + \gamma_3 > 0$ and $\gamma_2 + \gamma_3 \neq 0$ we have for a lattice sequence $\mathcal{G}^\varepsilon$ as in Definition 3.1.2*

$$\| C^{\mathcal{G}^\varepsilon}(f_1, f_2, f_3) \|_{\mathcal{C}^{\gamma_2+\gamma_3}_p(\mathcal{G}^\varepsilon, \rho_1\rho_2\rho_3)} \lesssim \|f_1\|_{\mathcal{C}^{\gamma_1}_p(\mathcal{G}^\varepsilon, \rho_1)} \|f_2\|_{\mathcal{C}^{\gamma_2}_\infty(\mathcal{G}^\varepsilon, \rho_2)} \|f_3\|_{\mathcal{C}^{\gamma_3}_\infty(\mathcal{G}^\varepsilon, \rho_3)} .$$

Further, property $(\mathscr{E})$ holds for $C^{\mathcal{G}^\varepsilon}$ (and the continuous commutator as in [GIP15, Lemma 2.4]) if the regularity on the left hand side is reduced by an arbitrary $\kappa > 0$.

Proof. The proof of the estimates works line-by-line as in [GP15c, Lemma 14] and the $(\mathscr{E})$-property follows as in Lemma 3.2.2 by exploiting that $\psi(\varepsilon \cdot) = 1$ on a ball of order ε^{-1}. □

3.3 Discrete diffusion operators

Let us construct a symmetric random walk on a Bravais lattice $\mathcal{G}^\varepsilon$ with mesh size ε which can reach every point (our construction follows [LL10]).

First we choose a subset of "jump directions" $\{g_1, \ldots, g_l\} \subseteq \mathcal{G} \backslash \{0\}$ such that $\mathbb{Z}g_1 + \ldots + \mathbb{Z}g_l = \mathcal{G}$ and a map $\kappa \colon \{g_1, \ldots, g_l\} \to (0, \infty)$. We then take as a rate for the jump from $z \in \mathcal{G}^\varepsilon$ to $z \pm \varepsilon g_i \in \mathcal{G}^\varepsilon$ the value $\kappa(g_i)/2\varepsilon^2$. In other words the generator of the random walk is

$$L^\varepsilon u(y) = \varepsilon^{-2} \sum_{e \in \{\pm g_i\}} \frac{\kappa(e)}{2} (u(y + \varepsilon e) - u(y)) , \tag{3.30}$$

which converges (for u nice enough) to $Lu = \frac{1}{2}\sum_{i=1}^{l} \kappa(g_i)\partial_{g_i}^2 u$ as ε tends to 0, where ∂_{g_i} denotes the directional derivative. In the case $\mathcal{G} = \mathbb{Z}^d$ and $\kappa(e_i) = 1/d$ we obtain the simple random walk with limiting generator $L = \frac{1}{2d}\Delta$. We can reformulate (3.30) by introducing a signed measure

$$\mu = \kappa(g_1)\left(\frac{1}{2}\delta_{g_1} + \frac{1}{2}\delta_{-g_1}\right) + \ldots + \kappa(g_l)\left(\frac{1}{2}\delta_{g_l} + \frac{1}{2}\delta_{-g_l}\right) - \sum_{i=1}^{l}\kappa(g_i)\delta_0\,,$$

which allows us to write $L^\varepsilon u = \varepsilon^{-2}\int_{\mathbb{R}^d} u(x+\varepsilon y)\mathrm{d}\mu(y)$ and $Lu = \frac{1}{2}\int_{\mathbb{R}^d} y \cdot \nabla^2 u\, y\,\mathrm{d}\mu(y)$. In fact we will also allow the random walk to have infinite range.

Definition 3.3.1. *We write $\mu \in \boldsymbol{\mu}(\omega) = \boldsymbol{\mu}(\omega, \mathcal{G})$ for $\omega \in \boldsymbol{\omega}$ if μ is a finite, signed measure on $\mathbb{R}^d$ with support on a Bravais lattice $\mathcal{G} \subseteq \mathbb{R}^d$ such that*

- $\langle \operatorname{supp} \mu \rangle = \mathcal{G}$,
- $\mu|_{\{0\}^c} \geqslant 0$,
- *for any $\lambda > 0$ we have $\int_{\mathbb{R}^d} e^{\lambda\omega(x)}\,\mathrm{d}|\mu|(x) = \int_{\mathcal{G}} e^{\lambda\omega(x)}\,\mathrm{d}|\mu|(x) < \infty$, where $|\mu|$ is the total variation of μ,*
- *$\mu(A) = \mu(-A)$ for (measurable) $A \subseteq \mathbb{R}^d$ and $\mu(\mathbb{R}^d) = 0$,*

where $\langle\cdot\rangle$ denotes the subgroup generated by $\cdot$ in $(\mathbb{R}^d, +)$. We associate a norm on $\mathbb{R}^d$ to $\mu \in \boldsymbol{\mu}(\omega)$ which is given by

$$\|x\|_\mu^2 = \frac{1}{2}\int_{\mathbb{R}^d} |x \cdot y|^2 \mathrm{d}\mu(y) = \frac{1}{2}\int_{\mathcal{G}} |x \cdot y|^2 \mathrm{d}\mu(y)\,.$$

We also write $\boldsymbol{\mu}(\boldsymbol{\omega}) := \bigcup_{\omega \in \boldsymbol{\omega}} \boldsymbol{\mu}(\omega)$.

Lemma 3.3.2. *The function $\|\cdot\|_\mu$ of Definition 3.3.1 is indeed a norm.*

Proof. The homogeneity is obvious and the triangle inequality follows from Minkowski's inequality. If $\|x\|_\mu = 0$ we have $x \cdot g = 0$ for all $g \in \operatorname{supp}\mu$. Since $\langle \operatorname{supp}\mu \rangle = \mathcal{G}$ we also have $x \cdot a_i = 0$ for the linearly independent vectors $a_1, \ldots, a_d$ from (3.1), which implies $x = 0$. □

Given $\mu \in \boldsymbol{\mu}(\boldsymbol{\omega})$ as in Definition 3.3.1 we can then generalize the formulas we found above.

Definition 3.3.3. *For $\mu \in \boldsymbol{\mu}(\omega)$ for $\omega \in \boldsymbol{\omega}$ as in Definition 3.3.1 and $\mathcal{G}^\varepsilon$ as in Definition 3.1.2 we set*

$$L^\varepsilon_\mu u(x) = \varepsilon^{-2} \int_{\mathcal{G}} u(x + \varepsilon y)\, \mathrm{d}\mu(y)$$

for $u \in \mathcal{S}'_\omega(\mathcal{G}^\varepsilon)$ and

$$(L_\mu u)\,(\varphi) := \frac{1}{2} \int_{\mathcal{G}} y \cdot \nabla^2 u \, y \, \mathrm{d}\mu(y)\,(\varphi) := \frac{1}{2} \int_{\mathcal{G}} y \cdot \nabla^2 u(\varphi)\, y \, \mathrm{d}\mu(y)$$

for $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}_\omega(\mathbb{R}^d)$. We write further $\mathscr{L}^\varepsilon_\mu, \mathscr{L}_\mu$ for the parabolic operators $\mathscr{L}^\varepsilon_\mu = \partial_t - L^\varepsilon_\mu$ and $\mathscr{L}_\mu = \partial_t - L_\mu$.

L^ε_μ is nothing but the infinitesimal generator of a random walk with sub-exponential moments (Lemma 3.5.5). By direct computation it can be checked that for $\mathcal{G} = \mathbb{Z}^d$ and with the extra condition $\int y_i y_j \mathrm{d}\mu(y) = 2\,\delta_{ij}$ we have the identities $\|\cdot\|_\mu = |\cdot|$ and $L_\mu = \Delta_{\mathbb{R}^d}$. In general L_μ is an elliptic operator with constant coefficients,

$$L_\mu u = \frac{1}{2} \int_{\mathcal{G}} y \cdot \nabla^2 u \, y \, \mathrm{d}\mu(y) = \frac{1}{2} \sum_{i,j} \int_{\mathcal{G}} y_i y_j \, \mathrm{d}\mu(y) \cdot \partial^{ij} u =: \frac{1}{2} \sum_{i,j} a^\mu_{ij} \cdot \partial^{ij} \varphi \,,$$

where (a^μ_{ij}) is a symmetric matrix. The ellipticity condition follows from the relation $x \cdot (a^\mu_{ij}) x = \|x\|^2_\mu$ and the equivalence of norms on $\mathbb{R}^d$. In terms of regularity we expect therefore that L^ε behaves like the Laplacian when we work on discrete spaces.

Lemma 3.3.4. *We have for $\gamma \in \mathbb{R}$, $p \in [1, \infty]$, $\omega \in \boldsymbol{\omega}$ and $\mu \in \boldsymbol{\mu}(\omega)$, $\rho \in \boldsymbol{\rho}(\omega)$*

$$\|L^\varepsilon_\mu u\|_{\mathcal{C}^{\gamma-2}_p(\mathcal{G}^\varepsilon, \rho)} \lesssim \|u\|_{\mathcal{C}^\gamma_p(\mathcal{G}^\varepsilon, \rho)} \,,$$

where $\mathcal{C}^\gamma_p(\mathcal{G}^\varepsilon, \rho) = \mathcal{B}^\gamma_{p,\infty}(\mathcal{G}^\varepsilon, \rho)$ is defined as in Definition 3.1.6.

For $\delta \in [0, 1]$ we further have

$$\|(L^\varepsilon_\mu - L_\mu)u\|_{\mathcal{C}^{\gamma-2-\delta}_p(\mathbb{R}^d, \rho)} \lesssim \varepsilon^\delta \|u\|_{\mathcal{C}^\gamma_p(\mathbb{R}^d, \rho)} \,,$$

where the action of L^ε_μ on $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$ should be read as

$$(L^\varepsilon_\mu u)(\varphi) = \varepsilon^{-2} \int_{\mathcal{G}} u\big(\varphi(\cdot - \varepsilon y)\big)\, \mathrm{d}\mu(y)$$

for $\varphi \in \mathcal{S}_\omega(\mathbb{R}^d)$. The involved constants are independent of ε

Proof. We start with the first inequality. With $\overline{\Psi}^{\mathcal{G}^\varepsilon,j} := \sum_{-1\leqslant i\leqslant j_{\mathcal{G}^\varepsilon}:\,|i-j|\leqslant 1}\Psi^{\mathcal{G}^\varepsilon,i} \in \mathcal{S}_\omega(\mathcal{G}^\varepsilon)$ we have by spectral support properties, similar as in the proof of Lemma 2.1.23, $\varDelta_j^{\mathcal{G}^\varepsilon}u = \overline{\Psi}^{j,\mathcal{G}^\varepsilon} *_{\mathcal{G}^\varepsilon} \varDelta_j^{\mathcal{G}^\varepsilon}u$. Via (3.17) we can read $\Psi^{j,\mathcal{G}^\varepsilon}$ and thus $\overline{\Psi}^{j,\mathcal{G}^\varepsilon}$ as a smooth function in $\mathcal{S}_\omega(\mathbb{R}^d)$ defined on all of $\mathbb{R}^d$. In this sense we read

$$\varDelta_j^{\mathcal{G}^\varepsilon}u = |\mathcal{G}^\varepsilon|\sum_{z\in|\mathcal{G}^\varepsilon|}\overline{\Psi}^j(\cdot - z)\,\varDelta_j^{\mathcal{G}^\varepsilon}u(z)\,, \tag{3.31}$$

as a smooth function on $\mathbb{R}^d$ in the following. Since μ integrates affine functions to zero we then have

$$\begin{aligned}\varDelta_j^{\mathcal{G}^\varepsilon}L_\mu^\varepsilon u(x) &= \varepsilon^{-2}\int_{\mathcal{G}}\mathrm{d}\mu(y)\,[\varDelta_j^{\mathcal{G}^\varepsilon}u(x+\varepsilon y) - \varDelta_j^{\mathcal{G}^\varepsilon}u(x) - \nabla(\varDelta_j^{\mathcal{G}^\varepsilon}u)(x)\cdot\varepsilon y]\\ &= \int_{\mathcal{G}}\mathrm{d}\mu(y)\int_0^1\mathrm{d}\zeta_1\int_0^1\mathrm{d}\zeta_2\;\, y\bullet\nabla^2(\varDelta_j^{\mathcal{G}^\varepsilon}u)(x+\varepsilon\zeta_1\zeta_2 y)y.\end{aligned}$$

Using (2.2) and the Minkowski inequality on the support of μ we then obtain

$$\begin{aligned}&\|\rho\varDelta_j^{\mathcal{G}^\varepsilon}L_\mu^\varepsilon u\|_{L^p(\mathcal{G}^\varepsilon)}\\ &\lesssim \int_{\mathcal{G}}\mathrm{d}\mu(y)\int_0^1\mathrm{d}\zeta_1\int_0^1\mathrm{d}\zeta_2 e^{\lambda\omega(\varepsilon\zeta_1\zeta_2 y)}|y|^2\left\|\rho(\cdot+\varepsilon\zeta_1\zeta_2 y)|\nabla^2(\varDelta_j^{\mathcal{G}^\varepsilon}u)(\cdot+\varepsilon\zeta_1\zeta_2 y)|\right\|_{L^p(\mathcal{G}^\varepsilon)},\end{aligned}$$

where λ is as in (2.2). By definition of $\boldsymbol{\mu}(\boldsymbol{\omega})$ we have

$$\int_0^1\mathrm{d}\zeta_1\int_0^1\mathrm{d}\zeta_2\int_{\mathcal{G}}\mathrm{d}\mu(y)\,|y|^2e^{\lambda\omega(\varepsilon\zeta_1\zeta_2 y)} \leqslant \int_0^1\mathrm{d}\zeta_1\int_0^1\mathrm{d}\zeta_2\int_{\mathcal{G}}\mathrm{d}\mu(y)\,|y|^2e^{\lambda\omega(y)} < \infty$$

so that we are left with the task to estimate

$$\begin{aligned}&\left\|\rho(\cdot+\varepsilon\zeta_1\zeta_2 y)|\nabla^2(\varDelta_j^{\mathcal{G}^\varepsilon}u)(\cdot+\varepsilon\zeta_1\zeta_2 y)|\right\|_{L^p(\mathcal{G}^\varepsilon)}\\ &\lesssim \|\nabla^2\overline{\Psi}^{\mathcal{G}^\varepsilon,j}(\cdot+\varepsilon\zeta_1\zeta_2)\|_{L^1(\mathcal{G}^\varepsilon,e^{\lambda\omega(\cdot+\varepsilon\zeta_1\zeta_2)})}\,\|\varDelta_j^{\mathcal{G}^\varepsilon}u\|_{L^p(\mathcal{G}^\varepsilon,\rho)}\,,\end{aligned}$$

where we applied (3.31) and Young's convolution inequality on $\mathcal{G}^\varepsilon$. Due to (3.17) and Lemma 3.1.7 we can estimate the first factor by 2^{j2} so that we obtain the total estimate

$$\|\varDelta_j^{\mathcal{G}^\varepsilon}L_\mu^\varepsilon u\|_{L^p(\mathcal{G}^\varepsilon,\rho)} \lesssim 2^{-j(\gamma-2)}\|u\|_{\mathcal{C}_p^\gamma(\mathcal{G}^\varepsilon,\rho)}$$

and the first estimate follows.

To show the second inequality we proceed essentially as above and use $\overline{\Psi}^j = \sum_{i:\,|i-j|\leqslant 1}\Psi^i$ where $\Psi^j = \mathcal{F}_{\mathbb{R}^d}^{-1}\varphi_j$ now really denotes the inverse transform of the partition on $\mathbb{R}^d$ as in (2.17) in Chapter 2. We then have $\Delta_j = \overline{\Psi}^j * \Delta_j$ so that one gets

$$\begin{aligned}\Delta_j(L^\varepsilon_\mu - L_\mu)u = &\int_0^1 \mathrm{d}\zeta_1 \int_0^1 \mathrm{d}\zeta_2 \int_{\mathcal{G}} \mathrm{d}\mu(y) \\ &\times \int_{\mathbb{R}^d} \mathrm{d}z\, y \bullet (\nabla^2 \overline{\Psi}^j(\cdot + \varepsilon\zeta_1\zeta_2 y - z) - \nabla^2\overline{\Psi}^j(\cdot - z)) y\, \Delta_j u(z)\,.\end{aligned}$$

As above we can then either get $2^{-j(\gamma-2)}\|u\|_{\mathcal{C}^\gamma_p(\mathcal{G}^\varepsilon,\rho)}$, by bounding each of the two second derivatives seperately, or $2^{-j(\gamma-3)}\varepsilon\|u\|_{\mathcal{C}^\gamma_p(\mathcal{G}^\varepsilon,\rho)}$, by exploiting the difference to introduce the third derivative. We obtain the second estimate by interpolation. □

3.3.1 Semigroup estimates

In Fourier space L^ε_μ can be represented by a Fourier multiplier

$$\mathcal{F}_{\mathcal{G}^\varepsilon}(L^\varepsilon_\mu u) = -l^\varepsilon_\mu \cdot \mathcal{F}_{\mathcal{G}^\varepsilon} u\,,$$

for $u \in \mathcal{S}'_\omega(\mathcal{G}^\varepsilon)$. The multiplier l^ε_μ is given by

$$l^\varepsilon_\mu(x) = -\int_{\mathcal{G}} \frac{e^{\imath\varepsilon 2\pi x\bullet y}}{\varepsilon^2}\,\mathrm{d}\mu(y) = \int_{\mathcal{G}} \frac{1-\cos(\varepsilon 2\pi x \bullet y)}{\varepsilon^2}\,\mathrm{d}\mu(y) = 2\int_{\mathcal{G}} \frac{\sin^2(\varepsilon\pi x\bullet y)}{\varepsilon^2}\,\mathrm{d}\mu(y)\,, \tag{3.32}$$

where we used that μ is symmetric with $\mu(\mathcal{G}) = \mu(\mathbb{R}^d) = 0$ and the trigonometric identity $1 - \cos = 2\sin^2$. The following lemma shows that l^ε is well defined as a multiplier (i.e. $l^\varepsilon \in C^\infty_\omega(\widehat{\mathcal{G}^\varepsilon})$). It is moreover the backbone of the semigroup estimates shown below.

Lemma 3.3.5. *For $\mu \in \boldsymbol{\mu}(\boldsymbol{\omega})$, $\sigma \in (0,1)$ the function l^ε_μ defined in (3.32) is an element of $\mathcal{S}_\omega(\widehat{\mathcal{G}^\varepsilon}) = C^\infty_\omega(\widehat{\mathcal{G}^\varepsilon})$ and satisfies*

- $|\partial^k l^\varepsilon_\mu(x)| \lesssim_\delta \varepsilon^{(|k|-2)\vee 0}(1+|x|^2)\delta^{|k|}(k!)^{1/\sigma}$ *for $\mu \in \boldsymbol{\mu}(\omega^{\exp}_\sigma)$, $\sigma \in (0,1)$ and any $\delta > 0$, $k \in \mathbb{N}^d$,*
- $l^\varepsilon_\mu \gtrsim_K |\cdot|^2$ *on every compact set $\varepsilon^{-1}K \subseteq \mathbb{R}^d$ with $K \cap \mathscr{R} = \{0\}$, where $\mathscr{R}$ is the reciprocal lattice of the* unscaled *lattice $\mathcal{G}$.*

Proof. We start by showing $|\partial^k l^\varepsilon_\mu(x)| \lesssim_\delta \varepsilon^{(|k|-2)\wedge 0}(1+|x|^2)\delta^{|k|}(k!)^{1/\sigma}$ which implies in particular $l^\varepsilon_\mu \in \mathcal{S}_\omega(\widehat{\mathcal{G}^\varepsilon})$ (the statement $l^\varepsilon_\mu \in \mathcal{S}_\omega(\widehat{\mathcal{G}^\varepsilon})$ for $\mu \in \boldsymbol{\mu}(\omega^{\mathrm{pol}})$ is again similar but easier and therefore omitted).

We study derivatives with $|k| = 0, 1$ first. We have

$$|l_\mu^\varepsilon(x)| = 2\left|\int_{\mathcal{G}} \frac{\sin^2(\varepsilon\pi x \cdot y)}{\varepsilon^2}\mathrm{d}\mu(y)\right| \lesssim \left|\int_{\mathcal{G}} \frac{\sin^2(\varepsilon\pi x \cdot y)}{|\varepsilon\pi x \cdot y|^2}|x \cdot y|^2 \mathrm{d}\mu(y)\right|$$
$$\lesssim \int_{\mathcal{G}} |y|^2 \mathrm{d}|\mu|(y) \cdot |x|^2 \lesssim |x|^2,$$

and

$$|\partial^i l_\mu^\varepsilon(x)|| \lesssim \int_{\mathcal{G}} \frac{|\sin(\varepsilon\pi x \cdot y)|}{|\varepsilon\pi x \cdot y|}|x||y|^2 \mathrm{d}|\mu|(y) \lesssim |x|\,.$$

For the higher derivatives we use that $\partial_x^k e^{\imath 2\pi\varepsilon x\cdot y} = (\imath\pi\varepsilon)^{|k|} y^k e^{\imath 2\pi\varepsilon x\cdot y}$ which gives (where $C > 0$ denotes as usual a changing constant)

$$|\partial^k l_\mu^\varepsilon(x)| \leqslant \varepsilon^{|k|-2} C^{|k|} \int_{\mathcal{G}} |y|^{|k|} \mathrm{d}|\mu|(y) \leqslant \varepsilon^{|k|-2} C^{|k|} \max_{x\in\mathbb{R}^d}(|x|^{|k|} e^{-\lambda|x|^\sigma}) \int_{\mathcal{G}} e^{\lambda|y|^\sigma} \mathrm{d}\mu(y)$$

for any $\lambda > 0$. Using $\max_{t\geqslant 0} t^a e^{-t} = a^a e^{-a}$ we end up with

$$|\partial^k l_\mu^\varepsilon(x)| \lesssim \varepsilon^{|k|-2} \frac{1}{\lambda^{|k|/\sigma}} C^{|k|} |k|^{|k|/\sigma} \lesssim \varepsilon^{|k|-2} \frac{1}{\lambda^{|k|/\sigma}} C^{|k|} (k!)^{1/\sigma}\,,$$

and our first claim follows by choosing $\lambda^{1/\sigma} := C/\delta$.

It remains to show that $l_\mu^\varepsilon/|\cdot|^2 \gtrsim 1$ on $\varepsilon^{-1}K$, which is equivalent to $l_\mu^1/|\cdot|^2 \gtrsim 1$ on K. We start by finding the zeros of l_μ^1 which, by periodicity can be reduced to finding all $x \in \widehat{\mathcal{G}}$ with $l_\mu^1(x) = 0$. But if $l_\mu^1(x) = 0$, then $y \cdot x \in \mathbb{Z}$ for any $y \in \operatorname{supp}\mu$, which gives with $\langle \operatorname{supp}\mu \rangle = \mathcal{G}$ that we must have $a_i \cdot x \in \mathbb{Z}$ for a_i as in (3.1). But since $x \in \widehat{\mathcal{G}}$ we have $x = x_1\hat{a}_1 + \ldots + x_d\hat{a}_d$ with $x_i \in [-1/2, 1/2)$ and $\hat{a}_i$ as in (3.2). Consequently

$$x_i = x \cdot a_i \in \mathbb{Z} \cap [-1/2, 1/2) = \{0\}\,,$$

and hence $x = 0$. The zero set of l_μ^1 is thus precisely the reciprocal lattice $\mathscr{R}$. By assumption $K \cap \mathscr{R} = \{0\}$ and it remains therefore to verify $l_\mu^1(x) \gtrsim |x|^2$ in an environment of 0 to finish the proof.

Note that there is in fact a finite subset $V \subseteq \operatorname{supp}\mu$ such that $\langle V \rangle = \mathcal{G}$ since only finitely many $y \in \operatorname{supp}\mu$ are needed to generate $a_1, \ldots, a_d$. We restrict ourselves to V:

$$l_\mu^1(x) = 2\int_{\mathcal{G}} \sin^2(\pi x \cdot y)\mathrm{d}\mu(y) \geqslant 2\int_V \sin^2(\pi x \cdot y)\mathrm{d}\mu(y)$$

For $x \in \widehat{\mathcal{G}}\backslash\{0\}$ small enough we can now bound $\int_V \sin^2(\pi x \cdot y)\mathrm{d}\mu(y) \gtrsim \int_V |x \cdot y|^2 \mathrm{d}\mu(y)$. The term on the right hand side defines a norm by the same arguments as in Lemma 3.3.2, and since it must be equivalent to $|\cdot|^2$ the proof is complete. □

Using that $\mathcal{S}_\omega(\widehat{\mathcal{G}^\varepsilon}) = C^\infty_\omega(\widehat{\mathcal{G}^\varepsilon})$ is stable under composition we can now define the Fourier multiplier

$$e^{tL^\varepsilon_\mu} f := \mathcal{F}^{-1}_{\mathcal{G}^\varepsilon}(e^{-tl^\varepsilon_\mu}\mathcal{F}_{\mathcal{G}^\varepsilon} f)$$

for $f \in \mathcal{S}'_\omega(\mathcal{G}^\varepsilon)$ which gives the (weak) solution to the problem $\mathscr{L}^\varepsilon g = 0$, $g(0) = f$. The regularizing effect of the semigroup is described in the following proposition.

Proposition 3.3.6. *We have for $\gamma \in \mathbb{R}, \beta \geqslant 0$, $p \in [1,\infty]$, $\omega \in \boldsymbol{\omega}$, $\mu \in \boldsymbol{\mu}(\omega)$ and $\rho \in \boldsymbol{\rho}(\omega)$*

$$\|e^{tL^\varepsilon_\mu} f\|_{\mathcal{C}^{\gamma+\beta}_p(\mathcal{G}^\varepsilon,\rho)} \lesssim t^{-\beta/2} \|f\|_{\mathcal{C}^\gamma_p(\mathcal{G}^\varepsilon,\rho)} \,, \tag{3.33}$$

$$\|e^{tL^\varepsilon_\mu} f\|_{\mathcal{C}^{\beta}_p(\mathcal{G}^\varepsilon,\rho)} \lesssim t^{-\beta/2} \|f\|_{L^p(\mathcal{G}^\varepsilon,\rho)} \,, \tag{3.34}$$

and for $\gamma \in (0,2)$

$$\|(e^{tL^\varepsilon_\mu} - \mathrm{Id}) f\|_{L^p(\mathcal{G}^\varepsilon,\rho)} \lesssim t^{\gamma/2} \|f\|_{\mathcal{C}^\gamma_p(\mathcal{G}^\varepsilon,\rho)} \,, \tag{3.35}$$

uniformly on compact intervals $t \in [0,T]$.

Proof. We show the claim for $\omega = \omega^{\exp}_\sigma$, $\sigma \in (0,1)$, the arguments for $\omega = \omega^{\mathrm{pol}}$ are similar but easier. Using spectral support properties we can rewrite for $j \geqslant -1$

$$\Delta_j e^{tL^\varepsilon_\mu} f = \mathcal{F}^{-1}_{\mathcal{G}^\varepsilon}\left(\sum_{i\colon |i-j|\leqslant 1} \varphi^{\mathcal{G}^\varepsilon}_i e^{-tl^\varepsilon_\mu} \cdot \mathcal{F}_{\mathcal{G}^\varepsilon} \Delta^{\mathcal{G}^\varepsilon}_j f\right) = \mathscr{K}_j(t,\cdot) *_{\mathcal{G}^\varepsilon} \Delta^{\mathcal{G}^\varepsilon}_j f \,,$$

where we set for $z \in \mathcal{G}^\varepsilon$

$$\mathscr{K}_j(t,z) := \int_{\widehat{\mathcal{G}^\varepsilon}} \mathrm{d}y\, e^{2\pi\imath z\cdot y} \sum_{i\colon |i-j|\leqslant 1} \varphi^{\mathcal{G}^\varepsilon}_i e^{-tl^\varepsilon_\mu}$$

Using the smear function ψ^ε from Section 3.1 we can rewrite this as an expression that is well-defined *for all $x \in \mathbb{R}^d$*

$$\mathscr{K}_j(t,x) := \int_{\mathbb{R}^d} \mathrm{d}y\, e^{2\pi\imath x\cdot y}\, \psi^\varepsilon(y) \sum_{i\colon |i-j|\leqslant 1} \left(\varphi^{\mathcal{G}^\varepsilon}_i\right)_{\mathrm{ext}}(y) \cdot e^{-tl^\varepsilon_\mu(y)} \,,$$

where we extended l^ε_μ (periodically) to all of $\mathbb{R}^d$ by relation (3.32). Consequently, we can apply Lemma 3.1.11 to give an expression for the scaled kernel

$$\mathscr{K}_{(j)}(t,x) := 2^{-jd}\mathscr{K}_j(t,2^{-j}x) = \int_{\mathbb{R}^d} \mathrm{d}y\, e^{2\pi\imath x\cdot y} \varphi_{(j)}(y) \cdot e^{-tl^\varepsilon_\mu(2^j y)} \,,$$

where we wrote $\varphi_{(j)} = \sum_{i\colon |i-j|\leqslant 1} \check{\phi}_{\langle i\rangle_\varepsilon}$ with $\check{\phi}_{\langle i\rangle_\varepsilon}$ as in Lemma 3.1.11.

Suppose we already know that for any $\lambda > 0$ and $x \in \mathcal{G}^\varepsilon$ the estimate

$$t^{\beta/2}|\mathscr{K}_{(j)}(t,x)| \lesssim e^{-\lambda|x|^\sigma} 2^{-j\beta} \tag{3.36}$$

holds. Then Young's inequality on $\mathcal{G}^\varepsilon$ in Lemma 3.1.7 shows (3.33) and (3.34) (for (3.34) we also need (3.16)). Using Lemma 3.3.7 below we can reduce the task of proving (3.36) to the simpler problem of proving the polynomial bound

$$t^{\beta/2}|x_i|^n|\mathscr{K}_{(j)}(t,x)| \lesssim_\delta \delta^n C^n (n!)^{1/\sigma} 2^{-j\beta}, \tag{3.37}$$

with a constant $C > 0$ and an arbitrarily small $\delta > 0$.

To show (3.37) we assume that $2^j\varepsilon \leqslant 1$, otherwise we are dealing with the scale $2^j \approx \varepsilon^{-1}$ and the arguments below can be easily modified. Integration by parts gives us

$$\begin{aligned} |x_i|^n|\mathscr{K}_{(j)}(t,x)| &= C^n \left| \int_{\mathbb{R}^d} \mathrm{d}y\, e^{2\pi\imath x\cdot y}\, \partial_{y_i}^n \left(\varphi_{(j)}(y) e^{-t l_\mu^\varepsilon(2^j y)} \right) \right| \\ &\leqslant C^n \int_{\mathbb{R}^d} \mathrm{d}y \left| \partial_{y_i}^n \left(\varphi_{(j)}(y) e^{-t 2^{2j} l_\mu^{2^j\varepsilon}(y)} \right) \right| . \end{aligned}$$

Now we have the following estimates for $k,\, m \in \mathbb{N}^d$, $n \in \mathbb{N}$

$$\begin{gathered} |\partial_i^k \varphi_{(j)}(y)| \lesssim \delta^k (k!)^{1/\sigma}, \qquad |\partial^m l_\mu^{\varepsilon 2^j}(y)| \lesssim \delta^m\, (m!)^{1/\sigma} \\ \left| (2^{2j}t)^{\beta/2} \left(e^{t2^{2j}\cdot} \right)^{(n)} (l_\mu^{2^j\varepsilon})(y) \right| \lesssim n^{n/\sigma}\delta^n\,, \end{gathered}$$

where we used that $\varphi_{(j)} \in C^\infty_{\omega,c}(\mathbb{R}^d)$ (with bounds that can be chosen independent of j by definition) and Lemma 3.3.5 with the assumption $2^j\varepsilon \leqslant 1$. Together with Leibniz's and Faà-di Bruno's formula (Lemma 3.5.7) and a lengthy but elementary calculation (3.37) follows and therefore also (3.36).

The last estimate (3.35) can be obtained as in the proof of Lemma [GP15b, Lemma 6.6] by using Lemma 3.5.4 below. □

Lemma 3.3.7. *Let $g\colon \mathbb{R}^d \to \mathbb{R}$, $\sigma > 0$ and $B > 0$. Suppose for any $\delta > 0$ there is a $C_\delta > 0$ such that for all $z \in \mathbb{R}^d$, $l \geqslant 0$ and $i = 1, \dots, d$*

$$|z_i^l g(z)| \lesssim_\delta \delta^l C_\delta^l (l!)^{1/\sigma} B\,.$$

It then holds for any $\lambda > 0$ and $z \in \mathbb{R}^d$

$$|g(z)| \lesssim_\lambda B e^{-\lambda|z|^\sigma}\,.$$

Proof. This follows ideas from [MW15, Proposition A.2]. Without loss of generality we can assume $|z| > 1$ (otherwise we get the required estimate by taking $l = 0$). Recall that we have $|z|^l \leqslant C^l \sum_{i=1}^d |z_i|^l$ where $C > 0$ denotes a constant that changes from line to line and is independent of l. Consequently, Stirling's formula gives

$$|e^{\lambda|z|^\sigma} g(z)| = \left|\sum_{k=0}^\infty \frac{\lambda^k}{k!}|z|^{\sigma k} g(z)\right| \lesssim \sum_{k=0}^\infty \frac{\lambda^k C^k}{k^k}|z|^{\lceil k\sigma\rceil}\,|g(z)| \lesssim \sum_{k=0}^\infty \frac{\lambda^k C^k}{k^k}\sum_{i=1}^d |z_i^{\lceil k\sigma\rceil} g(z)|$$
$$\lesssim B\sum_{k=0}^\infty \frac{\lambda^k C^k \delta^{k\sigma}}{k^k}\lceil k\sigma\rceil^{\lceil k\sigma\rceil/\sigma} \lesssim B\sum_{k=0}^\infty \frac{\lambda^k C^k \delta^{k\sigma}}{k^k} k^k = B\sum_{k=0}^\infty \lambda^k C^k \delta^{k\sigma} \lesssim_\lambda B\,,$$

where we used $\lceil k\sigma\rceil \leqslant k\lceil\sigma\rceil$ so that $\lceil k\sigma\rceil^{\lceil k\sigma\rceil/\sigma} \leqslant \lceil\sigma\rceil^{\frac{k}{\sigma}} k^k \leqslant C^k k^k$ and where we chose $\delta < (C\,\lambda)^{-\frac{1}{\sigma}}$ in the last step. □

3.4 Discrete Wick calculus

In this section we develop a general machinery for the use of discrete Wick contractions in the renormalization of discrete, singular SPDEs with i.i.d. noise which is completely analogous to the continuous Gaussian setting. Moreover, we build on the techniques of [CSZ17] to provide a criterion that identifies the scaling limits of discrete Wick products as multiple Wiener-Itô integrals. Our results are summarized in Lemma 3.4.1 and Lemma 3.4.2 below and although the use of these results is illustrated in this thesis only via the discrete parabolic Anderson model in Chapter 4 the approach extends in principle to any discrete formulation of popular singular SPDEs such as the KPZ equation or the Φ^4_d models.

Take a sequence of scaled Bravais lattices $\mathcal{G}^\varepsilon$ as in Definition 3.1.2. As a *discrete approximation to white noise* we take in this thesis independent (but *not necessarily identically distributed*) random variables $\big(\xi^\varepsilon(z)\big)_{z\in\mathcal{G}^\varepsilon}$ that satisfy

- $\mathbb{E}[\xi^\varepsilon(x)] = 0$,
- $\mathbb{E}\left[|\xi^\varepsilon(x)|^2\right] = |\mathcal{G}^\varepsilon|^{-1} = |\mathcal{G}|^{-1}\varepsilon^{-d}$,
- $\sup_{z\in\mathcal{G}^\varepsilon}\mathbb{E}\left[|\xi^\varepsilon(z)|^{p_\xi}\right] \lesssim \varepsilon^{-d/2\cdot p_\xi}$ for some $p_\xi \geqslant 2$.

For discrete multiple stochastic integrals with respect to the variables $\big(\xi^\varepsilon(z)\big)_{z\in\mathcal{G}^\varepsilon}$, that is for sums

$$\sum_{z_1,\dots,z_n\in\mathcal{G}^\varepsilon} f(z_1,\dots,z_n)\,\xi^\varepsilon(z_1)\dots\xi^\varepsilon(z_n)$$

with $f(z_1,\ldots,z_n)=0$ whenever $z_i=z_j$ for some $i\neq j$, it was shown in [CGP17, Proposition 4.3] that, if p_ξ is large enough, all moments can be bounded in terms of the ℓ^2 norm of f and the corresponding moments of the $(\xi^\varepsilon(z))_{z\in\mathcal{G}^\varepsilon}$. However, typically we will have to bound such expressions for more general f (which do not vanish on the diagonals) and in that case we first have to arrange our random variable into a finite sum of discrete multiple stochastic integrals in order to apply [CGP17, Proposition 4.3] for each of them. This arrangement can be done in several ways, here we follow [HS15] and regroup in terms of Wick polynomials.

Given random variables $(Y(j))_{z\in J}$ over some index set J and $I=(j_1,\ldots,j_n)\in J^n$ we set

$$Y^I=Y(j_1)\ldots Y(j_n)=\prod_{k=1}^n Y(j_k)$$

as well as $Y^\varnothing=1$. According to Definition 3.1 and Proposition 3.4 of [LM16], the *Wick product* $Y^{\diamond I}$ can be defined recursively by $Y^{\diamond\varnothing}:=1$ and

$$Y^{\diamond I}:=Y^I-\sum_{\varnothing\neq E\subset I}\mathbb{E}[Y^E]Y^{\diamond\, I\setminus E}\,. \tag{3.38}$$

For $I=(j_1,\ldots,j_n)\in J^n$ we also write

$$Y(j_1)\diamond\cdots\diamond Y(j_n):=Y^{\diamond I}\,.$$

By induction one easily sees that this product is commutative. In the case $j_1=\ldots=j_n$ we will write instead

$$Y(j_1)^{\diamond n}\,.$$

Lemma 3.4.1 (see also Proposition 4.3 in [CGP17]). *Let $\mathcal{G}^\varepsilon$ and $\big(\xi^\varepsilon(z)\big)_{z\in\mathcal{G}^\varepsilon}$ be as above, $n\geqslant 1$ and assume $p_\xi\geqslant 2n$. For $f\in L^2((\mathcal{G}^\varepsilon)^n)$ define the* discrete multiple stochastic integral *w.r.t $\big(\xi^\varepsilon(z)\big)$ by*

$$\mathscr{I}_n f:=\sum_{z_1,\ldots,z_n\in\mathcal{G}^\varepsilon}|\mathcal{G}^\varepsilon|^n f(z_1,\ldots,z_n)\,\xi^\varepsilon(z_1)\diamond\ldots\diamond\xi^\varepsilon(z_n)\,.$$

It then holds for $2\leqslant p\leqslant p_\xi/n$

$$\|\mathscr{I}_n f\|_{L^p(\mathbb{P})}\lesssim\|f\|_{L^2((\mathcal{G}^\varepsilon)^n)}\,.$$

Proof. In the following we silently identify $\mathcal{G}^\varepsilon$ with an enumeration by $\mathbb{N}$ so that we can write

$$\mathscr{I}_n f=\sum_{1\leqslant r\leqslant n,\,a\in A_r^n}r!\sum_{z_1<\ldots<z_r}|\mathcal{G}^\varepsilon|^n\tilde f_a(z_1,\ldots,z_r)\cdot\xi^\varepsilon(z_1)^{\diamond a_1}\diamond\ldots\diamond\xi^\varepsilon(z_r)^{\diamond a_r}\,,$$

where $A^n_r := \{a \in \mathbb{N}^r \,|\, \sum_i a_i = n\}$, $\tilde{f}_a$ denotes the symmetrized version of

$$f_a(z_1,\ldots,z_r) := f(\overbrace{z_1,\ldots,z_1}^{a_1\times},\ldots,\overbrace{z_r,\ldots,z_r}^{a_r\times})\cdot \mathbf{1}_{z_i\neq z_j\ \forall i\neq j}\,,$$

and where we used the independence of $\xi^\varepsilon(z_1),\ldots,\xi^\varepsilon(z_r)$ to decompose the Wick product (we did not show this property, but it is not hard to derive it from the definition of $\diamond$ we gave above). The independence and the zero mean of the Wick products allow us to see this as a sum of nested martingale transforms so that an iterated application of the Burkholder-Davis-Gundy inequality and Minkowski's inequality as in [CGP17, Proposition 4.3] gives the desired estimate

$$\begin{aligned}
\|\mathscr{I}_n f\|^2_{L^p(\mathbb{P})} &\lesssim \sum_{1\leqslant r\leqslant n,\, a\in A^n_r} \Big\| \sum_{z_1<\ldots<z_r} |\mathcal{G}^\varepsilon|^n \cdot \tilde{f}_a(z_1,\ldots,z_r)\cdot \xi^\varepsilon(z_1)^{\diamond a_1}\diamond\ldots\diamond\xi^\varepsilon(z_r)^{\diamond a_r}\Big\|^2_{L^p(\mathbb{P})} \\
&\lesssim \sum_{1\leqslant r\leqslant n,\, a\in A^n_r}\ \sum_{z_1<\ldots<z_r} |\mathcal{G}^\varepsilon|^{2n}\cdot|\tilde{f}_a(z_1,\ldots,z_r)|^2\cdot\prod_{j=1}^r \|\xi^\varepsilon(z_r)^{\diamond a_j}\|^2_{L^p(\mathbb{P})} \\
&\lesssim \sum_{1\leqslant r\leqslant n,\, a\in A^n_r}\ \sum_{z_1,\ldots,z_r} |\mathcal{G}^\varepsilon|^n|\tilde{f}_a(z_1,\ldots,z_r)|^2 \leqslant \|f\|^2_{L^2((\mathcal{G}^\varepsilon)^n)}\,,
\end{aligned}$$

where we used the bound $\|\xi^\varepsilon(z_r)^{\diamond a_j}\|^2_{L^p(\mathbb{P})} \lesssim |\mathcal{G}^\varepsilon|^{-a_j}$ which follows from (3.38) and our assumption on ξ^ε. □

Next, we provide a general criterion for the convergence of discrete multiple stochastic integrals to multiple Wiener-Itô integrals on $\mathbb{R}^d$. To this end we follow [CSZ17] and adapt their results to the Wick product setting of Lemma 3.4.1.

Lemma 3.4.2 (see also [CSZ17], Theorem 2.3)**.** *Let $\mathcal{G}^\varepsilon, n \in \mathbb{N}$ and $\big(\xi^\varepsilon(z)\big)$ be as in Lemma 3.4.1. For $k = 0,\ldots,n$ let $f^\varepsilon_k \in L^2((\mathcal{G}^\varepsilon)^k)$. We identify $(\mathcal{G}^\varepsilon)^k$ with a Bravais lattice in $k\cdot d$ dimensions via the orthogonal sum $(\mathcal{G}^\varepsilon)^k = \bigoplus_{i=1}^k \mathcal{G}^\varepsilon \subseteq \bigoplus_{i=1}^k \mathbb{R}^d = (\mathbb{R}^d)^k$ to define the Fourier transform $\mathcal{F}_{(\mathcal{G}^\varepsilon)^k} f^\varepsilon_k \in L^2((\widehat{\mathcal{G}^\varepsilon})^k)$ of f^ε_k. Assume that there exist $g_k \in L^2((\mathbb{R}^d)^k)$ with $|\mathbf{1}_{(\widehat{\mathcal{G}^\varepsilon})^k}\mathcal{F}_{(\mathcal{G}^\varepsilon)^k} f^\varepsilon_k| \leqslant g_k$ for all ε and $f_k \in L^2((\mathbb{R}^d)^k)$ such that $\lim_{\varepsilon\to 0}\|\mathbf{1}_{(\widehat{\mathcal{G}^\varepsilon})^k}\mathcal{F}_{(\mathcal{G}^\varepsilon)^k} f^\varepsilon_k - \mathcal{F}_{(\mathbb{R}^d)^k} f_k\|_{L^2((R^d)^k)} = 0$ for all $k \leqslant n$. Then the following convergence holds in distribution*

$$\sum_{k=0}^n \mathscr{I}_k f^\varepsilon_k \longrightarrow \sum_{k=0}^n \int_{(\mathbb{R}^d)^k} f_k(z_1,\ldots,z_k)\,\xi(\mathrm{d}z_1)\diamond\cdots\diamond\xi(\mathrm{d}z_k)\,,$$

where $\xi(\mathrm{d}z_1)\diamond\cdots\diamond\xi(\mathrm{d}z_k)$ indicates integration in the Wiener-Itô sense against the Gaussian measure induced by some white noise ξ on $\mathbb{R}^d$.

Remark 3.4.3. *The symbol $\diamond$ in integrals denotes in this thesis Skorohod integration. Note, that the integral above can indeed be read as an iterated Skorohod integral. We will encounter results similar to the one above when considering renormalizations of models in regularity structures in Chapter 7 below.*

Proof. We will write shorthand $\widehat{f_k^\varepsilon} := \mathcal{F}_{(\mathcal{G}^\varepsilon)^k} f_k^\varepsilon$ and $\widehat{f_k} = \mathcal{F}_{(\mathbb{R}^d)^k} f_k$.

This is a consequence of the results in [CSZ17]. For $z \in \mathcal{G}^\varepsilon$ let $G^\varepsilon(z) = z + [-\varepsilon/2, \varepsilon/2)a_1 + \ldots + [-\varepsilon/2, \varepsilon/2)a_d$, where $a_1, \ldots, a_d$ denote the vectors that span $\mathcal{G}$. For $x \in \mathbb{R}^d$ let $[x]_\varepsilon := z$ be the (unique) element $z \in \mathcal{G}^\varepsilon$ such that $x \in G^\varepsilon(z)$ and for $x \in (\mathbb{R}^d)^k$ set $[x]_\varepsilon = ([x_1]_\varepsilon, \ldots, [x_k]_\varepsilon)$. We will start by showing

$$\lim_{\varepsilon \to 0} \| f_k^\varepsilon([\cdot]_\varepsilon) - f_k \|_{L^2((\mathbb{R}^d)^k)} = 0 \tag{3.39}$$

for all k.

By Parseval's identity we have

$$\| f_k^\varepsilon([\cdot]_\varepsilon) - f_k \|_{L^2((\mathbb{R}^d)^k)} = \| \mathcal{F}_{(\mathbb{R}^d)^k}(f_k^\varepsilon([\cdot]_\varepsilon)) - \widehat{f_k} \|_{L^2((\mathbb{R}^d)^k)},$$

where $\mathcal{F}_{(\mathbb{R}^d)^k}$ denotes the Fourier transform on $(\mathbb{R}^d)^k$ for which one easily checks that

$$\mathcal{F}_{(\mathbb{R}^d)^k}(f_k^\varepsilon([\cdot]_\varepsilon)) = (\widehat{f_k^\varepsilon})_{\mathrm{ext}} \cdot p_k^\varepsilon,$$

where we recall that $(\widehat{f_k^\varepsilon})_{\mathrm{ext}}$ is the periodic extension of the discrete Fourier transform of f_k^ε (on $(\mathbb{R}^d)^k$) as in (3.10) and where

$$p_k^\varepsilon(y_1, \ldots, y_k) = \int_{G^1(0)^k} \frac{\mathrm{d}z_1 \ldots \mathrm{d}z_k}{|\mathcal{G}^1|^k} \, e^{-2\pi\imath\varepsilon(y_1 \cdot z_1 + \cdots + y_k \cdot z_k)}.$$

The function p_k^ε is uniformly bounded and tends to 1 as ε goes to 0. Now we apply Parseval's identity on $(\mathbb{R}^d)^k$ and once on $(\widehat{\mathcal{G}^\varepsilon})^k$ and obtain

$$\begin{aligned}\int_{(\mathbb{R}^d)^k} \mathrm{d}x_1 \ldots \mathrm{d}x_k \left| \big((\widehat{f_k^\varepsilon})_{\mathrm{ext}} \, p^\varepsilon\big)(x_1, \ldots, x_k) \right|^2 &= \sum_{z_1, \ldots, z_k \in \mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon|^k |f_k^\varepsilon(z_1, \ldots, z_k)|^2 \\ &= \int_{(\widehat{\mathcal{G}^\varepsilon})^k} \mathrm{d}x_1 \ldots \mathrm{d}x_k \left| \widehat{f_k^\varepsilon}(x_1, \ldots, x_k) \right|^2\end{aligned}$$

and thus

$$\begin{aligned}&\int_{((\widehat{\mathcal{G}^\varepsilon})^k)^c} \mathrm{d}x_1 \ldots \mathrm{d}x_k \left| \big((\widehat{f_k^\varepsilon})_{\mathrm{ext}} \, p^\varepsilon\big)(x_1, \ldots, x_k) \right|^2 \\ &= \int_{(\widehat{\mathcal{G}^\varepsilon})^k} \mathrm{d}x_1 \ldots \mathrm{d}x_k \big(|\widehat{f_k^\varepsilon}|^2 (1 - |p^\varepsilon|^2) \big)(x_1, \ldots, x_k) \, .\end{aligned}$$

Since $\mathbf{1}_{(\widehat{\mathcal{G}^\varepsilon})^k}\widehat{f}_k^\varepsilon$ is uniformly in ε bounded by the $L^2((\mathbb{R}^d)^k)$ function g_k and since $1-|p^\varepsilon|^2$ converges pointwise to zero, it follows from the dominated convergence theorem that $\mathbf{1}_{((\widehat{\mathcal{G}^\varepsilon})^k)^c}(\widehat{f}_k^\varepsilon)_{\mathrm{ext}}\,p_k^\varepsilon$ converges to zero in $L^2((\mathbb{R}^d)^k)$. Thus, we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\|(\widehat{f}_k^\varepsilon)_{\mathrm{ext}}\,p_k^\varepsilon-\widehat{f}_k\|_{L^2((\mathbb{R}^d)^k)}&=\lim_{\varepsilon\to 0}\|\mathbf{1}_{(\widehat{\mathcal{G}^\varepsilon})^k}\widehat{f}_k^\varepsilon p_k^\varepsilon-\widehat{f}_k\|_{L^2((\mathbb{R}^d)^k)}\\ &\leqslant\lim_{\varepsilon\to 0}\|(\mathbf{1}_{(\widehat{\mathcal{G}^\varepsilon})^k}\widehat{f}_k^\varepsilon-\widehat{f}_k)p_k^\varepsilon\|_{L^2((\mathbb{R}^d)^k)}+\lim_{\varepsilon\to 0}\|\widehat{f}_k(1-p_k^\varepsilon)\|_{L^2((\mathbb{R}^d)^k)}=0,\end{aligned}$$

where for the first term we used that p_k^ε is uniformly bounded in ε and that by assumption $\mathbf{1}_{(\widehat{\mathcal{G}^\varepsilon})^k}\widehat{f}_k^\varepsilon$ converges to $\widehat{f}_k$ in $L^2((\mathbb{R}^d)^k)$ and for the second term we combined the fact that p_k^ε converges pointwise to 1 with the dominated convergence theorem. We have therefore shown (3.39). Note that this implies

$$\|f_k^\varepsilon([\cdot]_\varepsilon)\mathbf{1}_{\forall i\neq j\,[z_i]_\varepsilon\neq[z_j]_\varepsilon}-f_k\|_{L^2(\mathbb{R}^d)}\to 0\qquad\&\qquad\|f_k^\varepsilon([\cdot]_\varepsilon)\mathbf{1}_{\exists i\neq j\,[z_i]_\varepsilon=[z_j]_\varepsilon}\|_{L^2(\mathbb{R}^d)}\to 0\,.\tag{3.40}$$

As in the proof of Lemma 3.4.1 we identify $\mathcal{G}^\varepsilon$ with some arbitrary enumeration $\mathbb{N}\to\mathcal{G}^\varepsilon$ and use the set $A_r^k=\{a\in\mathbb{N}^r\mid\sum_i a_i=k\}$ so that we can write

$$\mathscr{I}_k f_k^\varepsilon=\sum_{1\leqslant r\leqslant k,\,a\in A_r^k}r!\sum_{z_1<\ldots<z_r}|\mathcal{G}^\varepsilon|^k\tilde{f}_{\varepsilon,a}^k(z_1,\ldots,z_r)\cdot\prod_{j=1}^r\xi^\varepsilon(z_j)^{\diamond a_j}\,,$$

where we denote as in the proof of Lemma 3.4.1 by $\tilde{f}_{\varepsilon,a}^k$ the symmetrized restriction of f_ε^k to $(\mathbb{R}^d)^r$. By Theorem 2.3 of [CSZ17] we see that the $r=k$ term of $\mathscr{I}_k f_k^\varepsilon$ converges due to 3.40 to the desired limit in distribution, so that we only have to show that the remainding terms vanish as ε tends to 0. The idea is to redefine the noise in these terms by $\overline{\xi}_j^\varepsilon(z)=\xi^\varepsilon(z)^{\diamond a_j}/r_j^\varepsilon(z)$ where $r_j^\varepsilon(z):=\sqrt{\mathrm{Var}(\xi^\varepsilon(z)^{\diamond a_j})\cdot|\mathcal{G}^\varepsilon|}\lesssim|\mathcal{G}^\varepsilon|^{(1-a_j)/2}$, so that in view of [CSZ17, Lemma 2.3] it suffices to show that

$$\sum_{z_1<\ldots<z_r}|\mathcal{G}^\varepsilon|^r\prod_{j=1}^r r_j^\varepsilon(z_j)^2\cdot|\tilde{f}_{k,a}^\varepsilon(z_1,\ldots,z_r)|^2\lesssim\sum_{z_1<\ldots<z_r}|\mathcal{G}^\varepsilon|^k\cdot|\tilde{f}_{k,a}^\varepsilon(z_1,\ldots,z_r)|^2\to 0\,,$$

but this follows from (3.40). □

3.5 Technical Results

Lemma 3.5.1. *Given a lattice $\mathcal{G}$ as in* (3.1) *we denote the translations of the closed parallelotope $G:=[0,1]a_1+\ldots+[0,1]a_d$ by $\mathbb{G}:=\{g+G\,|\,g\in\mathcal{G}\}$. Let $\Omega\subseteq\mathcal{G}$ and set $\overline{\Omega}:=\bigcup_{G'\in\mathbb{G},\,G'\cap\Omega\neq\varnothing}G'$. If for a measurable function $f:\overline{\Omega}\to\mathbb{R}_+$ there is a $c\geqslant 1$*

such that for any $g \in \Omega$ there is a $G'(g) \in \mathbb{G}$, $g \in G'(g)$ with $f(g) \leqslant c \cdot \operatorname{ess\,inf}_{x\in G'} f(x)$ then it also holds

$$\sum_{g\in\Omega} |\mathcal{G}| f(g) \leqslant c \cdot 2^d \int_{\overline{\Omega}} f(x)\mathrm{d}x\,.$$

Proof. Indeed

$$\begin{aligned}\sum_{g\in\Omega} |\mathcal{G}| f(g) &\leqslant c \sum_{g\in\Omega} \int_{G'(g)} f(x)\mathrm{d}x \leqslant c \sum_{g\in\Omega} \sum_{G'\in\mathbb{G},\, g\in G'} \int_{G'(g)} f(x)\mathrm{d}x \\ &= c \sum_{G'\in\mathbb{G},\, G'\subseteq\overline{\Omega}} \; \sum_{g\in\Omega, g\in G'} \int_{G'} f(x)\mathrm{d}x \overset{(\triangle)}{=} 2^d c \sum_{G'\in\mathbb{G}} \int_{G'} f(x)\mathrm{d}x = 2^d c \int_{\overline{\Omega}} f(x)\mathrm{d}x\,,\end{aligned}$$

where we used in $(\triangle)$ that the d-dimensional parallelotope has 2^d vertices. □

Lemma 3.5.2. *The mappings $(\mathcal{F}_{\mathcal{G}}, \mathcal{F}_{\mathcal{G}}^{-1})$ as defined in subsection 3.1.1 map the spaces $(\mathcal{S}_\omega(\mathcal{G}), \mathcal{S}_\omega(\widehat{\mathcal{G}}))$ and $(\mathcal{S}'_\omega(\mathcal{G}), \mathcal{S}'_\omega(\widehat{\mathcal{G}}))$ to each other.*

Proof. We only consider the non-standard case $\omega = \omega_\sigma^{\exp}$, $\sigma \in (0,1)$. Given $f \in \mathcal{S}_\omega(\mathcal{G})$ the sequence

$$\mathcal{F}_{\mathcal{G}} f(x) = |\mathcal{G}| \sum_{k\in\mathcal{G}} f(k) e^{2\pi\imath k\cdot x}$$

does obviously converge to a smooth function that is periodic on $\widehat{\mathcal{G}}$. We estimate on $\widehat{\mathcal{G}}$ (and thus on every compact set)

$$\left|\partial^\alpha \sum_{k\in\mathcal{G}} |\mathcal{G}| f(k) e^{2\pi\imath k\cdot x}\right| \lesssim_\lambda \sum_{k\in\mathcal{G}} |\mathcal{G}||k|^{|\alpha|} e^{-\lambda|k|^\sigma}$$

We can use Lemma 3.5.1 for $|\cdot|^{|\alpha|} e^{-\lambda|\cdot|^{1/s}}$ with $\Omega = \mathcal{G}$ and $c > 0$ of the form $c = C(\lambda) \cdot C^{|\alpha|}$ (C denoting a positive constant that may change from line to line) which yields

$$\left|\partial^\alpha \sum_{k\in\mathcal{G}} |\mathcal{G}| f(k) e^{2\pi\imath k\cdot x}\right| \lesssim_\lambda C^{|\alpha|} \int_{\mathbb{R}^d} |x|^{|\alpha|} e^{-\lambda|x|^\sigma}\mathrm{d}x$$

We now proceed as in [Hör05, Lemma 12.7.4] and estimate the integral by the $\Gamma-$function

$$\begin{aligned}&\int_{\mathbb{R}^d} |x|^{|\alpha|} e^{-\lambda|x|^\sigma}\mathrm{d}x \lesssim \int_0^\infty r^{|\alpha|+d-1} e^{-\lambda r^\sigma}\mathrm{d}r \lesssim_\lambda \lambda^{-s|\alpha|} \int_0^\infty r^{|\alpha|+d-1} e^{-r^\sigma}\mathrm{d}r \\ &\lesssim \lambda^{-|\alpha|/\sigma} \Gamma((|\alpha|+d-1)/\sigma) \overset{\text{Stirling}}{\lesssim} \lambda^{-|\alpha|/\sigma} C^{|\alpha|} |\alpha|^{|\alpha|/\sigma}\,.\end{aligned}$$

Since we can choose $\lambda > 0$ arbitrarily large we see that indeed $f \in C^\infty_\omega(\widehat{\mathcal{G}})$.

For the opposite direction, $f \in \mathcal{S}_\omega(\widehat{\mathcal{G}})$, we use that by integration by parts for $z \in \mathcal{G}$, $l \geqslant 0$, $i = 1, \dots, d$ $\left|z_i^l \cdot \mathcal{F}_{\mathcal{G}}^{-1} f(z)\right| \lesssim C^l \sup_{\widehat{\mathcal{G}}} (\partial^i)^l f \lesssim C^l \varepsilon^l l^{l/\sigma}$ With Stirling's formula and Lemma 3.3.7 we then obtain $\left|\mathcal{F}_{\mathcal{G}}^{-1} f(z)\right| \lesssim e^{\lambda |z|^\sigma}$. This shows the statement for the pair $(\mathcal{S}_\omega(\mathcal{G}), \mathcal{S}_\omega(\widehat{\mathcal{G}}))$. The estimates above show that $\mathcal{F}_{\mathcal{G}}, \mathcal{F}_{\mathcal{G}}^{-1}$ are in fact continuous w.r.t to the corresponding topologies so that the statement for the dual spaces $(\mathcal{S}'_\omega(\mathcal{G}), \mathcal{S}'_\omega(\widehat{\mathcal{G}}))$ immediately follows. □

Lemma 3.5.3 (Mixed Young inequality)**.** *For $f\colon \mathbb{R}^d \to \mathbb{C}$ and $g\colon \mathcal{G} \to \mathbb{C}$, for which this is defined, we set for $x \in \mathbb{R}^d$*

$$f *_{\mathcal{G}} g(x) := \sum_{k \in \mathcal{G}} |\mathcal{G}| f(x-k) g(k)$$

Then for $r, p, q \in [1, \infty]$ with $1 + 1/r = 1/p + 1/q$

$$\|f *_{\mathcal{G}} g\|_{L^r(\mathbb{R}^d)} \leqslant \sup_{x \in \mathbb{R}^d} \|f(x - \cdot)\|_{L^p(\mathcal{G})}^{1-\frac{p}{r}} \cdot \|f\|_{L^p(\mathbb{R}^d)}^{\frac{p}{r}} \|g\|_{L^q(\mathcal{G})} .$$

(with the convention $1/\infty = 0$, $\infty/\infty = 1$).

Proof. We assume $p, q, r \in (1, \infty)$. The remaining cases are easy to check.

The proof is based on Hölder's inequality on $\mathcal{G}$ with $\frac{1}{r} + \frac{1}{\frac{rp}{r-p}} + \frac{1}{\frac{rq}{r-q}} = 1$

$$\begin{aligned}
|f *_{\mathcal{G}} g(x)| &\leqslant \sum_{k \in \mathcal{G}} |\mathcal{G}| \left(|f(x-k)|^p |g(k)|^q\right)^{1/r} \cdot |f(x-k)|^{\frac{r-p}{r}} |g(k)|^{\frac{r-q}{r}} \\
&\overset{\text{Hölder}}{\leqslant} \left\| \left(|f(x-\cdot)|^p |g(\cdot)|^q\right)^{1/r} \right\|_{L^r(\mathcal{G})} \\
&\times \left\| |f(x-\cdot)|^{\frac{r-p}{r}} \right\|_{L^{\frac{rp}{r-p}}(\mathcal{G})} \cdot \left\| |g(\cdot)|^{\frac{r-q}{r}} \right\|_{L^{\frac{rq}{r-q}}(\mathcal{G})} \\
&= \left(\sum_{k \in \mathcal{G}} |\mathcal{G}| (|f(x-k)|^p |g(k)|^q \right)^{1/r} \sup_{x' \in \mathbb{R}^d} \|f(x' - \cdot)\|_{L^p(\mathcal{G})}^{\frac{r-p}{r}} \|g\|_{L^q(\mathcal{G})}^{\frac{r-q}{r}}
\end{aligned}$$

Raising this expression to the rth power and integrating shows the claim. □

Lemma 3.5.4. *For $t \geqslant 0$, $p \in [1, \infty]$, $\omega \in \boldsymbol{\omega}$, $\rho \in \boldsymbol{\rho}(\omega)$ and $\mu \in \boldsymbol{\mu}(\omega)$ we have on compact time intervals for $\mathcal{G}^\varepsilon$ as in Definition 3.1.2*

$$\|e^{tL_\mu^\varepsilon} \varphi\|_{L^p(\mathcal{G}^\varepsilon, \rho)} \lesssim \|\varphi\|_{L^p(\mathcal{G}^\varepsilon, \rho)} .$$

and for $\beta > 0$

$$\|e^{tL_\mu^\varepsilon} \varphi\|_{L^p(\mathcal{G}^\varepsilon, \rho)} \lesssim t^{-\beta/2} \|\varphi\|_{\mathcal{C}_p^{-\beta}(\mathcal{G}^\varepsilon, \rho)}$$

uniformly in ε.

Proof. With the random walk $(X_t^\varepsilon)_{t\in\mathbb{R}_+}$ which is generated by L_μ^ε on $\mathcal{G}^\varepsilon$ and starts from $0 \in \mathcal{G}^\varepsilon$ we can express the semigroup as $e^{tL_\mu^\varepsilon} f(x) = \mathbb{E}[f(x + X_t^\varepsilon)]$ so that by Jensen's inequality

$$\sum_{x\in\mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon| \left|\rho(x) e^{tL_\mu^\varepsilon} f(x)\right|^p \leqslant \mathbb{E}\left[\sum_{x\in\mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon| \, |\rho(x) f(x + X_t^\varepsilon)|^p\right]$$
$$\overset{(2.2)}{\lesssim} \mathbb{E}\left[e^{p\lambda\omega(X_t^\varepsilon)} \sum_{x\in\mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon| \, |f(x + X_t^\varepsilon)\rho(x + X_t^\varepsilon)|^p\right] = \mathbb{E}\left[e^{\lambda p\omega(X_t^\varepsilon)}\right] \|f\|^p_{L^p(\mathcal{G}^\varepsilon,\rho)} \,.$$

where $\lambda > 0$ is as in (2.2). Application of the next lemma finishes the proof of the first estimate. The second estimate follows as in Lemma 6.6. of [GP15b]. $\square$

Lemma 3.5.5. *The random walk X^ε generated by L^ε on $\mathcal{G}^\varepsilon$ satisfies for any $c, c' > 0$ and $t \in [0, T]$*

$$\mathbb{E}[e^{c\omega(|X_t^\varepsilon|)}] \lesssim_{c,c'} e^{c'\omega(t)} \,.$$

Proof. We assume $\omega = \omega_\sigma^{\exp}$ for $\sigma \in (0, 1)$, if $\omega = \omega^{\mathrm{pol}}$ the proof follows by similar, but simpler arguments. We write shorthand $s = 1/\sigma$.

By the Lévy-Khintchine-formula we have for $\theta \in \mathbb{R}$ $\mathbb{E}[e^{\imath\theta X_t^\varepsilon}] = e^{-t/\varepsilon^2 \int_{\mathcal{G}} (1 - e^{\imath\theta\varepsilon x}) \mathrm{d}\mu(x)} = e^{-tl^\varepsilon(\theta)}$. We want to bound first for $k \geqslant 1$

$$\mathbb{E}[|X_{t,1}^\varepsilon|^k + \ldots + |X_{t,d}^\varepsilon|^k] = \sum_{i=1}^d \left|\partial_{\theta_i}^k|_{\theta=0} \mathbb{E}[e^{\imath\theta X_t^\varepsilon}]\right|$$

To this end we apply Faá-di-Brunos formula (Lemma 3.5.7) with $u(v) = e^{-tv}$, $v(\theta) = l^\varepsilon(\theta)$ Note that with Lemma 3.3.5

$$u^{(m)}(1) = (-t)^m$$
$$|\partial_{\theta_i}^{\alpha_i} v(0)| \lesssim_\delta \delta^{|\alpha_i|} (\alpha_i!)^s$$

Thus with $A_{m,k} = \{\alpha \in \mathbb{N}_{>0}^m \mid \sum_i \alpha_i = k\}$ for some $\delta \in (0, 1]$

$$\left|\partial_{\theta_i}^k|_{\theta=0} \mathbb{E}[e^{\imath\theta X_t^\varepsilon}]\right| = \left|\sum_{1\leqslant m\leqslant k,\, \alpha\in A_{m,k}} \frac{k!}{m!\alpha!} u^{(m)}(1) \prod_{i=1}^m \partial_{\theta_i}^{\alpha_i} v(0)\right|$$
$$\lesssim \sum_{1\leqslant m\leqslant k,\, \alpha\in A_{m,k}} \frac{k!}{m!\alpha!} t^m \prod_{i=1}^m (\alpha_i!)^s \delta^{|\alpha_i|} \leqslant \delta^k \sum_{1\leqslant m\leqslant k,\, \alpha\in A_{m,k}} t^m k! (m!)^{s-1} \prod_{i=1}^m (\alpha_i!)^{s-1}$$
$$\overset{\text{Lemma 3.5.6}}{\leqslant} \delta^k (k!)^s \sum_{1\leqslant m\leqslant k,\, \alpha\in A_{m,k}} t^m = \delta^k (k!)^s \sum_{1\leqslant m\leqslant k} \binom{k-1}{m-1} t^m$$
$$= \delta^k (k!)^s t (1+t)^{k-1} \leqslant \delta^k (k!)^s (1+t)^k$$

With $|x|_k^k := |x_1|^k + \ldots + |x_d|^k$ we get

$$\mathbb{E}[|X_t^\varepsilon|_k^k] \lesssim \delta^k (k!)^s (1+t)^k$$

and therefore, using Stirling's formula and $|x|^k \lesssim C^k \cdot |x|_k^k$ (with a generic constant $C > 0$ as usual),

$$\begin{aligned}
\mathbb{E}[e^{c|X_t^\varepsilon|^\sigma}] &\lesssim 1 + \mathbb{E}[e^{c|X_t^\varepsilon|^\sigma} \mathbf{1}_{|X_t^\varepsilon| \geqslant 1}] \\
&\leqslant 1 + \sum_{n=0}^{\infty} \frac{c^n}{n!} \mathbb{E}[|X_t^\varepsilon|^{\lceil n\sigma \rceil}] \lesssim 1 + \sum_{n=0}^{\infty} \frac{C^n t^{\lceil n\sigma \rceil}}{n^n} \delta^{\lceil n\sigma \rceil} \lceil n\sigma \rceil^{\lceil n\sigma \rceil s} \\
&\lesssim 1 + t \sum_{n=0}^{\infty} \frac{C^n \delta^{n\sigma} t^{n\sigma}}{n^n} n^n = 1 + t e^{C\delta^\sigma t^\sigma}
\end{aligned}$$

Choosing $\delta > 0$ small enough finishes the proof. □

Lemma 3.5.6. *We have for $j \in \mathbb{N}_{>0}$ and $\alpha_1, \ldots, \alpha_j \in \mathbb{N}_{>0}$*

$$j!\alpha_1! \ldots \alpha_j! \leqslant (\alpha_1 + \ldots + \alpha_j)!$$

Proof. This follows from a simple combinatorical argument: Let $k = \alpha_1 + \ldots + \alpha_j$. Then while the right hand side corresponds to the number of arbitrary orderings of k elements, the left hand side corresponds to the number of possibilities to arrange these elements while keeping them together in sets of size $\alpha_1, \ldots, \alpha_j$. □

Lemma 3.5.7 (Faà di Bruno's formula as stated in [CS96])**.** *For multiindices ν, $\mu \in \mathbb{N}^d$ we write $\mu \prec \nu$ if one of the two cases holds*

1. *$|\mu| < |\nu|$*
2. *$|\mu| = |\nu|$ and there is a $i \in \{1, \ldots, d\}$ such that $\mu^i < \nu^i$ and $\mu^j = \nu^j$ for $j < i$.*

For $\lambda \in \mathbb{N}\backslash\{0\}$ and $\nu \in \mathbb{N}^d$ and $s \in \mathbb{N}$ with $1 \leqslant s \leqslant |\nu|$ introduce the set

$$\begin{aligned}
p_s(\nu, \lambda) = \Big\{ (\boldsymbol{\lambda}; \boldsymbol{\mu}) = (\lambda^1, \ldots, \lambda^s; \mu^1, \ldots, \mu^d) \in (\mathbb{N}\backslash\{0\})^s \times (\mathbb{N}^d)^s \\
\mid 0 \prec \mu^1 \prec \ldots \prec \mu^1, \sum_{i=1}^s \lambda^i = \lambda, \sum_{i=1}^s \lambda^i \mu^i = \nu \Big\}
\end{aligned}$$

For smooth $g : \mathbb{R}^d \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ we have for $\nu \in \mathbb{N}^d \backslash \{0\}$

$$\partial^\nu (f(g)) = \sum_{\lambda=1}^{|\nu|} f^{(n)}(g) \cdot \sum_{s=1}^{|\nu|} \sum_{(\boldsymbol{\lambda}; \boldsymbol{\mu}) \in p_s(\nu, \lambda)} \nu! \prod_{i=1}^d \frac{1}{\lambda_i! \mu_i!} \left(\partial^{\mu_i} g\right)^{\lambda_i}.$$

Chapter 4

Weak universality of the parabolic Anderson model

With the theory presented in Chapter 3 at hand we can analyze stochastic models on unbounded lattices using paracontrolled techniques. We here prove a weak universality result for the linear parabolic Anderson model, which is stated in Theorem 4.3.6 below. Most of the content presented here is taken, with minor adaptions, from [MP17].

For $F \in C^2(\mathbb{R};\mathbb{R})$ with $F(0) = 0$ and bounded second derivative we consider the equation

$$\mathscr{L}^1_\mu \phi^\varepsilon = F(\phi^\varepsilon) \cdot \eta^\varepsilon, \qquad \phi^\varepsilon(0) = |\mathcal{G}|^{-1} \mathbf{1}_{\cdot=0} \tag{4.1}$$

on $\mathbb{R}_+ \times \mathcal{G}$, where $\mathcal{G} \subseteq \mathbb{R}^2$ is a two-dimensional Bravais lattice, $\mathscr{L}^1_\mu = \partial_t - L^1_\mu$ is some discrete diffusion operator on the lattice $\mathcal{G}$ as described in Definition 3.3.3, induced by some $\mu \in \boldsymbol{\mu}(\omega)$ with $\omega = \omega^{\exp}_\sigma$ for some $\sigma \in (0,1)$ (the upper index 1 indicates that we *did not scale* the lattice $\mathcal{G}$ yet). The family $(\eta^\varepsilon(z))_{z\in\mathcal{G}} \in \mathcal{S}'_\omega(\mathcal{G})$ consists of independent (not necessarily identically distributed) random variables satisfying

$$\mathbb{E}[\eta^\varepsilon] = -F'(0) c^\varepsilon_\mu \varepsilon^2\,, \qquad \mathrm{Var}(\eta^\varepsilon) = \frac{1}{|\mathcal{G}|}\,\varepsilon^2\,,$$

where $c^\varepsilon_\mu > 0$ is a constant of order $O(|\log \varepsilon|)$ which we will fix in Section 4.2 below. We further assume that for every ε and $z \in \mathcal{G}$ the variable $\eta^\varepsilon(z)$ has moments of order $p_\xi > 14$ such that

$$\mathbb{E}\big[|\eta^\varepsilon(z) - \mathbb{E}[\eta^\varepsilon(z)]|^{p_\xi}\big] \lesssim \varepsilon^{p_\xi}\,.$$

The lower bound 14 for p_ξ might seem quite arbitrary at the moment, we will explain this choice in Remark 4.3.1 below.

Note that η^ε is of order $O(\varepsilon)$ while its expectation is of order $O(\varepsilon^2|\log\varepsilon|)$, so we are considering a small shift away from the "critical" expectation 0.

We are interested in the behaviour of (4.1) for large scales in time and space. Setting $u^\varepsilon(t,x) := \varepsilon^{-2}\phi^\varepsilon(\varepsilon^{-2}t, \varepsilon^{-1}x)$ and $\xi^\varepsilon(x) := \varepsilon^{-2}(\eta^\varepsilon(\varepsilon^{-1}x) + F'(0)c_\mu^\varepsilon\varepsilon^2)$ modifies the problem to

$$\mathscr{L}_\mu^\varepsilon u^\varepsilon = F^\varepsilon(u^\varepsilon)(\xi^\varepsilon - F'(0)c_\mu^\varepsilon), \qquad u^\varepsilon(0) = |\mathcal{G}^\varepsilon|^{-1}\mathbf{1}_{\cdot=0}\,, \tag{4.2}$$

where $u^\varepsilon\colon \mathbb{R}_+ \times \mathcal{G}^\varepsilon \to \mathbb{R}$ is defined on refining lattices $\mathcal{G}^\varepsilon$ in $d=2$ as in Definition 3.1.2 and where $F^\varepsilon := \varepsilon^{-2}F(\varepsilon^2\cdot)$. The potential $(\xi^\varepsilon(x))_{x\in\mathcal{G}^\varepsilon}$ is scaled such that it satisfies for $x \in \mathcal{G}^\varepsilon$

- $\mathbb{E}[\xi^\varepsilon(x)] = 0$,
- $\mathbb{E}\left[|\xi^\varepsilon(x)|^2\right] = |\mathcal{G}^\varepsilon|^{-1} = |\mathcal{G}|^{-1}\varepsilon^{-2}$,
- $\sup_{z\in\mathcal{G}^\varepsilon}\mathbb{E}\left[|\xi^\varepsilon(z)|^{p_\xi}\right] \lesssim \varepsilon^{-p_\xi}$ for some $p_\xi > 14$.

and is thus a discrete approximation to white noise in dimension 2 as in Section 3.4.

Consequently, we expect $\mathscr{E}^\varepsilon\xi^\varepsilon$ to converge in distribution to white noise on $\mathbb{R}^2$, we will see in Lemma 4.2.3 below that this is indeed the case. In Theorem 4.3.6 we show that $\mathscr{E}^\varepsilon u^\varepsilon$ converges in distribution to the solution u of the linear parabolic Anderson model on $\mathbb{R}^2$,

$$\mathscr{L}_\mu u = F'(0)u(\xi - F'(0)\infty), \qquad u(0) = \delta, \tag{4.3}$$

where ξ is white noise on $\mathbb{R}^2$, δ is the Dirac delta distribution, "$-\infty$" denotes some renormalization and $\mathscr{L}_\mu$ is the limiting operator from Definition 3.3.3. The existence and uniqueness of a solution to (4.3) were first established in [HL15] (for more regular initial conditions) by using a "partial Cole-Hopf transformation" which turns the equation into a well-posed PDE, an argument we will also apply in Chapter 8 below. Using the continuous versions of the objects defined in Chapter 3 we can modify the arguments of [GIP15] to give an alternative proof of their result, see Corollary 4.3.5 below. The limit of (4.2) only sees $F'(0)$ and forgets the structure of the non-linearity F, so in that sense the linear parabolic Anderson model arises as a universal scaling limit.

Let us illustrate this result with a (far too simple) model: Suppose F is of the form $F(\phi) = \phi(1-\phi)$ and let us first consider

$$\partial_t\phi = \eta\cdot F(\phi), \qquad \phi(0) \in (0,1)\,,$$

for some $\eta \in \mathbb{R}$. If $\eta > 0$, then ϕ describes the evolution of the concentration of a growing population in a pleasant environment, which however shows some saturation effects represented by the factor $(1-\phi)$ in the definition of F. For $\eta < 0$ the individuals live in unfavorable conditions, say in competition with a rival species. From this perspective equation (4.1) describes the dynamics of a population that migrates between diverse habitats. The meaning of our universality result is that if we tune down the random potential η^{ε} and counterbalance the growth of the population with some renormalization (think of a death rate), then from far away we can still observe its growth (or extinction) without feeling any saturation effects.

The analysis of (4.2) and the convergence proof are based on the lattice version of paracontrolled distributions that was presented in Chapter 3. As a first step we discuss in Section 4.1 the Schauder theory for the operator $\mathscr{L}^{\varepsilon}_{\mu}$. Since our aim is to start our system in the quite irregular Dirac delta distribution (or rather its discrete analogue $|\mathcal{G}^{\varepsilon}|^{-1}\mathbf{1}_{\cdot=0}$) we expect a blow-up for the solution in the considered norms at $t=0$ so that we first introduce singular spaces. In order to apply the Schauder theory for paracontrolled distributions one needs a nice interplay of the used operator with the paraproduct. We introduce for this purpose a modified paraproduct $\lll$ as in [GIP15]. In Section 4.2 we study the convergence of the stochastic data, such as $\mathscr{E}^{\varepsilon}\xi^{\varepsilon}$. The main result of this chapter is then formulated in Theorem 4.3.6 of Section 4.3.

While Section 4.3 is devoted to the study of the problem described above, Sections 4.1 and 4.2 are formulated in a general set-up and in particular for any dimension d.

4.1 Schauder estimates

In Chapter 3 we only considered distributions $f \in \mathcal{S}'(\mathcal{G})$ with spatial dependence. When considering discrete approximations to SPDEs of parabolic type such as (4.1) we want our solution to be defined on

$$[0,T] \times \mathcal{G}\,,$$

As the initial condition is usually more irregular than the solution it is further necessary to allow for a possible blowup around 0. We follow here closely [GP15b] and introduce for this purpose time-weighted *parabolic spaces* $\mathscr{L}^{\nu,\gamma}_{p,T}$.

Definition 4.1.1. *Given $\nu \geqslant 0$, $T>0$ and a family of increasing normed spaces $X = (X(s))_{s\in[0,T]}$ we define the space*

$$\mathcal{M}^{\nu}_{T}X := \left\{ f\colon [0,T] \to X(T) \,\middle|\, \|f\|_{\mathcal{M}^{\nu}_{T}X} = \sup_{t\in[0,T]} \|t^{\nu} f(t)\|_{X(t)} < \infty \right\}\,,$$

and for $\gamma \in (0,1)$

$$C_T^\gamma X = \left\{ f \in C([0,T], X(T)) \,\middle|\, \|f\|_{C_T^\gamma X} < \infty \right\} ,$$

where

$$\|f\|_{C_T^\gamma X} = \sup_{t \in [0,T]} \|f(t)\|_{X(t)} + \sup_{0 \leqslant s \leqslant t \leqslant T} \frac{\|f(s) - f(t)\|_{X(t)}}{|s-t|^\gamma} .$$

For a lattice $\mathcal{G}$, $\nu \geqslant 0, T > 0, \gamma \in (0,2)$ *and a pointwise decreasing map* $\rho \colon [0,T] \ni t \mapsto \rho(t) \in \boldsymbol{\rho}(\omega)$ *for some* $\omega \in \boldsymbol{\omega}$ *we set*

$$\mathscr{L}_{p,T}^{\nu,\gamma}(\mathcal{G}, \rho) = \left\{ f \colon [0,T] \to \mathcal{S}'_\omega(\mathcal{G}) \,\middle|\, \|f\|_{\mathscr{L}_{p,T}^{\nu,\gamma}(\mathcal{G},\rho)} < \infty \right\} ,$$

where

$$\|f\|_{\mathscr{L}_{p,T}^{\nu,\gamma}(\mathcal{G},\rho)} = \|t \mapsto t^\nu f(t)\|_{C_T^{\gamma/2} L^p(\mathcal{G},\rho)} + \|f\|_{\mathcal{M}_T^\nu \mathcal{C}_p^\gamma(\mathcal{G},\rho)} .$$

We define the continuous analogue $\mathscr{L}_{p,T}^{\nu,\gamma}(\mathbb{R}^d, \rho)$ *in the same manner.*

Note that ρ in $\mathscr{L}_{p,T}^{\nu,\gamma}$ now really is a *time-dependent* weight. Whenever we take some fixed $\rho \in \boldsymbol{\rho}(\boldsymbol{\omega})$ as an argument of $\mathscr{L}_{p,T}^{\nu,\gamma}$ we identify it with the constant function $[0,T] \ni t \mapsto \rho$.

Standard arguments show that if X is a sequence of increasing Banach spaces with decreasing norms, such as $L^p(\mathcal{G},\rho)$ or $\mathcal{C}_p^\gamma(\mathcal{G},\rho)$ with decreasing weight $\rho(t)$, all the spaces in the previous definition are in fact complete in their (semi-)norms. At least for $\nu = 0$, $\gamma \in (0,1)$ and $p = \infty$ we can easily give an alternative characterization of the parabolic space $\mathscr{L}_{p,T}^{\nu,\gamma}(\mathbb{R}^d, \rho)$ in terms of the anisotropic space-time Besov spaces from Definition 2.1.24, namely

$$\mathscr{L}_{\infty,T}^{0,\gamma}(\mathbb{R}^d, \rho) = \mathcal{C}_{\mathfrak{s}_{\mathrm{par}}}^\gamma([0,T] \times \mathbb{R}^d, \rho)$$

where $\mathfrak{s}_{\mathrm{par}} = (2, 1, \ldots, 1) \in \mathbb{R}^{d+1}$ is the parabolic scaling vector. In this chapter will work with polynomial weights:

$$\langle x \rangle^{-\kappa} = (1 + |x|^2)^{-\kappa/2} \in \boldsymbol{\rho}(\omega^{\mathrm{pol}}) \subseteq \bigcap_{\sigma \in (0,1)} \boldsymbol{\rho}(\omega_\sigma^{\exp})$$

for $\kappa > 0$ and sub-exponential weights

$$e_{l+t}^\sigma(x) = e^{-(l+t)(1+|x|)^\sigma} \in \boldsymbol{\rho}(\omega_\sigma^{\exp})$$

for $\sigma \in (0,1)$, $l \in \mathbb{R}$ and a parameter $t \geqslant 0$ which later we will identify with a time variable. This choice was inspired by [HL15], the only difference is that they consider $\sigma = 1$ which is not permitted for us as explained in Remark 2.1.3. There is no deeper reason why we picked the smoothened polynomial weight $\langle x \rangle^{-\kappa}$ instead of, say, $(1+|x|)^{-\kappa}$. However, in Chapter 8 below this choice will turn out convenient, so that for the sake of rigidity we take the same weight here. The non-smooth choice for the sub-exponential weight will shorten some proofs below.

We now study the Schauder estimates for $\mathscr{L}_\mu^\varepsilon$ in terms of the spaces introduced in Definition 4.1.1. Let us introduce

$$I_\mu^\varepsilon f(t) = \int_0^t e^{(t-s)L_\mu^\varepsilon} f(s)\,\mathrm{d}s \tag{4.4}$$

The notation $\mathscr{L}_{p,T}^{\nu,\gamma}(\mathcal{G}, e_l^\sigma)$ in the following Lemma means that we take the time-dependent weight $(e_{l+t}^\sigma)_{t\in[0,T]}$, while $e_l^\sigma\langle x\rangle^{-\kappa}$ stands for the time-dependent weight $(e_{l+t}^\sigma\langle x\rangle^{-\kappa})_{t\in[0,T]}$.

Lemma 4.1.2. *Let $\mathcal{G}^\varepsilon$ be as in Definition 3.1.2 and let $\gamma \in (0,2), \nu \in [0,1), p \in [1,\infty]$, $\sigma \in (0,1)$, $\mu \in \boldsymbol{\mu}(\omega_\sigma^e)$ and $T > 0$. If $\beta \in \mathbb{R}$ is such that $(\gamma+\beta)/2 \in [0,1)$, then we have uniformly in ε*

$$\|s \mapsto e^{sL_\mu^\varepsilon} f_0\|_{\mathscr{L}_{p,T}^{(\gamma+\beta)/2,\gamma}(\mathcal{G}^\varepsilon, e_l^\sigma)} \lesssim \|f_0\|_{\mathcal{C}_p^{-\beta}(\mathcal{G}^\varepsilon, e_l^\sigma)}\,, \tag{4.5}$$

and if $\kappa \geqslant 0$ is such that $\nu+\kappa/\sigma \in [0,1)$, $\gamma + 2\kappa/\sigma \in (0,2)$ also

$$\|I_\mu^\varepsilon f\|_{\mathscr{L}_{p,T}^{\nu,\gamma}(\mathcal{G}^\varepsilon, e_l^\sigma)} \lesssim \|f\|_{\mathcal{M}_T^\nu \mathcal{C}_p^{\gamma+2\kappa/\sigma-2}(\mathcal{G}^\varepsilon, e_l^\sigma\langle x\rangle^{-\kappa})}\,. \tag{4.6}$$

The involved constants are independent of ε.

Proof. The proof is along the lines of Lemma 6.6 in [GP15b] with the use of the simple estimate

$$e_{l+t}^\sigma(x) \lesssim \frac{1}{|t-s|^{\kappa/\sigma}}\langle x\rangle^{-\kappa} e_{l+s}^\sigma(x), \qquad t \geqslant s,$$

which is similar to an inequality from the proof of Proposition 4.2 in [HL15] and the reason for the appearance of the term $2\,\kappa/\sigma$ in the lower estimate (the factor 2 arises due to the parabolic construction of our spaces). We need $\nu+\kappa/\sigma \in [0,1)$ so that the singularity $|t-s|^{-\nu-\kappa/\sigma}$ is integrable on $[0,t]$. $\square$

For the comparison of the parabolic spaces $\mathscr{L}_{p,T}^{\nu,\gamma}$ the following lemma will be convenient.

Lemma 4.1.3. *For $\mathcal{G}^\varepsilon$ as in Definition 3.1.2, $\gamma \in (0,2)$, $\nu \in (0,1)$, $\varepsilon \in [0,\gamma\wedge 2\nu)$, $p \in [1,\infty]$, $T > 0$ and a pointwise decreasing $\mathbb{R}_+ \ni s \mapsto \rho(s) \in \boldsymbol{\rho}(\omega)$ for some $\omega \in \boldsymbol{\omega}$ we have*

$$\|f\|_{\mathscr{L}_{p,T}^{\nu-\varepsilon/2,\gamma-\varepsilon}(\mathcal{G}^\varepsilon,\rho)} \lesssim \|f\|_{\mathscr{L}_{p,T}^{\nu,\gamma}(\mathcal{G}^\varepsilon,\rho)}\,,$$

and for $\nu \in [0,1)$ *and* $\varepsilon \in (0,\gamma)$

$$\|f\|_{\mathscr{L}^{\nu,\gamma-\varepsilon}_{p,T}(\mathcal{G}^\varepsilon,\rho)} \lesssim \mathbf{1}_{\nu=0}\|f(0)\|_{\mathcal{C}^{\gamma-\varepsilon}_p(\mathcal{G}^\varepsilon,\rho)} + T^{\varepsilon/2}\|f\|_{\mathscr{L}^{\nu,\gamma}_{p,T}(\mathcal{G}^\varepsilon,\rho)}\,.$$

The involved constants are independent in ε.

Proof. The first estimate is proved as in [GP15b, Lemma 6.8]. For $\nu = 0$ the proof of the second inequality works as in Lemma 2.11 of [GP15b]. The general case follows from the fact that $f \in \mathscr{L}^{\nu,\gamma}_{p,T}$ if and only if $t \mapsto t^\nu f \in \mathscr{L}^{0,\gamma}_{p,T}$. □

4.1.1 The modified paraproduct

In order to apply the Schauder estimates for operators such as $\mathscr{L}^\varepsilon_\mu$ in the context of paraproducts $\prec$ it turns out [GIP15] that it is essential to have a commutation property, in the sense that for $f_1, f_2 : [0,T] \to \mathcal{S}'_\omega(\mathcal{G}^\varepsilon)$ the difference

$$\mathscr{L}^\varepsilon_\mu(f_1 \prec f_2) - f_1 \prec \mathscr{L}^\varepsilon_\mu f_2$$

is of a better regularity then the single terms of this difference. Since $\mathscr{L}^\varepsilon_\mu = \partial_t - L^\varepsilon_\mu$ involves a time derivative and the paraproduct is a pure construction in the space variable (at least in Chapter 3 which we are refering to) there is no reason why this should be true. We follow therefore [GIP15] in introducing a (discrete) modified paraproduct $\lll$ instead.

Definition 4.1.4. *Fix a function* $\varphi^{\lll} \in C^\infty_c((0,\infty);\mathbb{R}_+)$ *such that* $\int_{\mathbb{R}} \varphi^{\lll}(s)\mathrm{d}s = 1$. *We then set*

$$Q_j f(t) := \int_{-\infty}^t 2^{2jd}\varphi^{\lll}(2^{2j}(t-s))f(s\vee 0)\mathrm{d}s, \qquad j \geqslant -1\,.$$

and define for a Bravais lattice $\mathcal{G}$ *and* $\omega \in \boldsymbol{\omega}$ *the* discrete modified paraproduct *as*

$$f_1 \lll^{\mathcal{G}} f_2 := \sum_{-1\leqslant j_1,j_2\leqslant j_{\mathcal{G}}:\, j_1<j_2-1} Q_{j_2}\Delta^{\mathcal{G}}_{j_1} f_1 \cdot \Delta^{\mathcal{G}}_{j_2} f_2$$

for f_1, $f_2\colon \mathbb{R}_+ \to \mathcal{S}'_\omega(\mathcal{G})$ *for which this expression is well defined. We may drop the index* $\mathcal{G}$ *if there is no risk for confusion.*

As in [GP15b] we silently identify f_1 in $f_1 \lll f_2$ with $t \mapsto f_1(t)\mathbf{1}_{t>0}$ if $f_1 \in \mathcal{M}^\nu_T\mathcal{C}^\gamma_p$. In [GIP15, GP15b] the object $f_1 \lll f_2$ was defined for f_1, $f_2\colon \mathbb{R}_+ \to \mathcal{S}'(\mathbb{R}^d)$, the generalization to the ultra-distribution case $f_1, f_2\colon \mathbb{R}_+ \to \mathcal{S}'_\omega(\mathbb{R}^d)$ is however now obvious. Note that we really only take the Fourier transform in space $\mathbb{R}^d$ and only use some arbitrary mollifiers build from $\varphi^{\lll}$ to deal with the time variable. This is due to the fact that the time variable will only be taken in some compact interval

$[0, T]$. The paraproduct $\lll$ does now indeed have the desired commutation property stated above, compare Lemma 4.1.7 below. In Chapter 5 and 6 below we propose a different and more general method that works with *space-time* paraproducts instead, the corresponding commutation result is formulated in Theorem 6.1.9 in Chapter 6. In order to overcome the obstacle posed by the finite time time interval we there make use of extensions instead of introducing cut-off functions such as $\varphi^{\lll}$.

The discrete modified paraproduct introduced in Definition 4.1.4 allows for similar estimates as in Lemma 3.2.2.

Lemma 4.1.5. *Let $\mathcal{G}^\varepsilon$ be as in Definition 3.1.2, $\beta \in \mathbb{R}$, $p \in [1, \infty]$, $\nu \in [0, 1)$, $t > 0, \gamma < 0$ and let $\rho_1, \rho_2 \colon \mathbb{R}_+ \to \boldsymbol{\rho}(\omega)$ for some $\omega \in \boldsymbol{\omega}$ with ρ_1 pointwise decreasing.*

$$t^\nu \|f_1 \lll f_2(t)\|_{\mathcal{C}_p^{\gamma+\beta}(\mathcal{G}^\varepsilon, \rho_1(t)\rho_2(t))} \lesssim \|f_1\|_{\mathcal{M}_t^\nu \mathcal{C}_p^\gamma(\mathcal{G}^\varepsilon, \rho_1)} \|f_2(t)\|_{\mathcal{C}^\beta(\mathcal{G}^\varepsilon, \rho_2(t))}$$

and

$$t^\nu \|f_1 \lll f_2(t)\|_{\mathcal{C}_p^{\beta}(\mathcal{G}^\varepsilon, \rho_1(t)\rho_2(t))} \lesssim \|f_1\|_{\mathcal{M}_t^\nu L^p(\mathcal{G}^\varepsilon, \rho_1)} \|f_2(t)\|_{\mathcal{C}^\beta(\mathcal{G}^\varepsilon, \rho)} .$$

Both estimates have the property $(\mathscr{E})$ if the regularity on the left hand side is decreased by an arbitrary $\kappa > 0$. The involved constants are independent of ε.

Proof. The proof is the same as for [GP15b, Lemma 6.4]. Property $(\mathscr{E})$ is shown as in Lemma 3.2.2. □

We further have an estimate in terms of the parabolic spaces $\mathscr{L}_{p,T}^{\nu,\gamma}(\mathcal{G}, \rho)$ we introduced in Definition 4.1.1.

Lemma 4.1.6. *We have for $\gamma \in (0, 2)$, $p \in [1, \infty]$, $\nu \in [0, 1)$ and pointwise decreasing $\rho_1, \rho_2 \colon \mathbb{R}_+ \to \boldsymbol{\rho}(\omega)$, for some $\omega \in \boldsymbol{\omega}$, the estimate*

$$\|f_1 \lll f_2\|_{\mathscr{L}_{p,T}^{\nu,\gamma}(\mathcal{G}^\varepsilon, \rho_1\rho_2)} \lesssim \|f_1\|_{\mathscr{L}_{p,T}^{\nu,\delta}(\mathcal{G}^\varepsilon, \rho_1)} \left(\|f_2\|_{C_T\mathcal{C}^\gamma(\mathcal{G}^\varepsilon, \rho_2)} + \|\mathscr{L}_\mu^\varepsilon f_2\|_{C_T\mathcal{C}^{\gamma-2}(\mathcal{G}^\varepsilon, \rho_2)}\right)$$

for any $\delta > 0$ and any diffusion operator $\mathscr{L}_\mu^\varepsilon$ induced by some $\mu \in \boldsymbol{\mu}(\omega)$ as in Definition 3.3.3 below. The involved constant is independent of ε.

Proof. The proof is as in [GP15b, Lemma 6.7] and uses Lemma 4.1.7 below. □

We now finally prove the announced commutation property between the discrete modified paraproduct $\lll$ and generators of random walks on Bravais lattices.

Lemma 4.1.7. *For $\gamma \in (0, 2)$, $\beta \in \mathbb{R}$, $p \in [1, \infty]$, $\nu \in [0, 1)$ and $\rho_1, \rho_2 \colon \mathbb{R}_+ \to \boldsymbol{\rho}(\omega)$ for some $\omega \in \boldsymbol{\omega}$, with ρ_1 pointwise decreasing, we have for $t > 0$*

$$t^\nu \|(f_1 \lll f_2 - f_1 \prec f_2)(t)\|_{\mathcal{C}_p^{\gamma+\beta}(\mathcal{G}^\varepsilon, \rho_1(t)\rho_2(t))} \lesssim \|f_1\|_{\mathscr{L}_{p,t}^{\nu,\gamma}(\mathcal{G}^\varepsilon, \rho_1)} \|f_2(t)\|_{\mathcal{C}^\beta(\mathcal{G}^\varepsilon, \rho_2(t))}$$

and

$$t^\nu \|\mathscr{L}_\mu^\varepsilon(f_1 \ll f_2) - f_1 \ll \mathscr{L}_\mu^\varepsilon f_2)(t)\|_{\mathcal{C}_p^{\gamma+\beta-2}(\mathcal{G}^\varepsilon, \rho_1(t)\rho_2(t))} \lesssim \|f_1\|_{\mathscr{L}_{p,t}^{\nu,\gamma}(\mathcal{G}^\varepsilon,\rho_1)} \|f_2(t)\|_{\mathcal{C}^\beta(\mathcal{G}^\varepsilon,\rho_2(t))} .$$

where $\mathscr{L}_\mu^\varepsilon = \partial_t - L_\mu^\varepsilon$ *is a discrete diffusion operator induced by some* $\mu \in \boldsymbol{\mu}(\omega)$ *as in Definition 3.3.3. The involved constants are independent of* ε*.*

Proof. Again we can almost follow along the lines of the proof in [GP15b, Lemma 6.5] with the only difference that in the derivation of the second estimate the application of the "product rule" of $\mathscr{L}_\mu^\varepsilon$ does not yield a term $-2\nabla f \ll \nabla f_2$ but a more complex object, namely

$$\int_{\mathbb{R}^d} \frac{\mathrm{d}\mu(y)}{\varepsilon^2}\; D_y^\varepsilon f_1 \ll D_y^\varepsilon f_2 \,, \tag{4.7}$$

where $D_y^\varepsilon f_1(t,x) = f_1(t, x+\varepsilon y) - f_1(t,x)$ and similar for f_2. The bound on (4.7) follows from Lemma 4.1.5 once we can show

$$\|D_y^\varepsilon \phi\|_{\mathcal{C}_p^{\gamma-1}(\mathcal{G}^\varepsilon,\rho_1)} \lesssim \|\phi\|_{\mathcal{C}_p^{\gamma}(\mathcal{G}^\varepsilon,\rho_1)} |y|\varepsilon \tag{4.8}$$

for any $\gamma \in \mathbb{R}$. Note that due to Lemma 3.1.11 we can write

$$\varDelta_j D_\varepsilon^y \phi = \left(\tilde{\Psi}^{\varepsilon,j}_{\cdot+\varepsilon y} - \tilde{\Psi}^{\varepsilon,j}\right) *_{\mathcal{G}^\varepsilon} \phi \,,$$

where $\tilde{\Psi}^{\varepsilon,j} = \mathscr{E}^\varepsilon \Psi^{\mathcal{G}^\varepsilon,j} = 2^{jd}\phi_{\langle j\rangle_\varepsilon}(2^j\cdot)$ with $\phi_{\langle j\rangle_\varepsilon} \in \mathcal{S}_\omega(\mathbb{R}^d)$. With

$$\tilde{\Psi}^{\varepsilon,j}_{x+\varepsilon y} - \tilde{\Psi}^{\varepsilon,j}_x = 2^j \int_0^1 2^{jd}\phi_{\langle j\rangle_\varepsilon}(2^j(x+\zeta\varepsilon y))\,\mathrm{d}\zeta \cdot y\varepsilon$$

we get (4.8) by applying Lemma 3.1.7.

□

Morally, the reason why the diffusion operator $\mathscr{L}_\mu^\varepsilon$ can be pulled on the second factor f_2 in the product $f_1 \ll f_2$ is that f_2 describes the small-scale behavior of this object which is the regime where $\mathscr{L}_\mu^\varepsilon$ acts.

4.2 Convergence of the stochastic data

Let $\mathcal{G}^\varepsilon$ be, as in Definition 3.1.2, a sequence of refining lattices build from some Bravais lattice $\mathcal{G}$ in dimension d. Let further $\big(\xi^\varepsilon(z)\big)_{z\in\mathcal{G}^\varepsilon}$ be a discrete approximation to white noise on $\mathcal{G}^\varepsilon$ as in Section 3.4.

Fix a symmetric $\chi \in C^{\infty}_{\omega,c}(\mathbb{R}^2)$, independent of ε, which is 0 on $\frac{1}{4}\cdot\widehat{\mathcal{G}}$ and 1 *outside* of $\frac{1}{2}\cdot\widehat{\mathcal{G}}$ and define for $\mu \in \boldsymbol{\mu}(\boldsymbol{\omega})$

$$Y^{\varepsilon}_{\mu} := \mathcal{F}^{-1}_{\mathcal{G}^{\varepsilon}}\left(\frac{\chi}{l^{\varepsilon}_{\mu}}\,\mathcal{F}_{\mathcal{G}^{\varepsilon}}\xi^{\varepsilon}\right).$$

where l^{ε}_{μ} is as in (3.32) the multiplier of the diffusion operator $\mathscr{L}^{\varepsilon}_{\mu}$ associated to μ. Note that $\mathscr{L}^{\varepsilon}_{\mu}Y^{\varepsilon}_{\mu} = -L^{\varepsilon}_{\mu}Y^{\varepsilon}_{\mu} = \chi(D_{\mathcal{G}^{\varepsilon}})\xi^{\varepsilon} := \mathcal{F}^{-1}_{\mathcal{G}^{\varepsilon}}(\chi\cdot\mathcal{F}_{\mathcal{G}^{\varepsilon}}\xi^{\varepsilon})$ so that Y^{ε}_{μ} is a time independent solution to the heat (or Poisson) equation on $\mathcal{G}^{\varepsilon}$ induced by our operator $\mathscr{L}^{\varepsilon}_{\mu}$. Note that χ is not scaled by ε and only serves as a cut-off for the "Fourier modes" around 0, where $\frac{1}{l^{\varepsilon}_{\mu}}$ diverges.

Our first task will be to measure the regularity of the sequences (ξ^{ε}), (Y^{ε}_{μ}) in the discrete Besov spaces introduced in Subsection 3.1.2.

As an application of the discrete Wick calculus we introduced in Section 3.4 we can bound the moments of ξ^{ε} and Y^{ε}_{μ} in Besov spaces. We also want to control the resonant product $Y^{\varepsilon}_{\mu}\circ\xi^{\varepsilon}$, for which we introduce the renormalization constant

$$c^{\varepsilon}_{\mu} := \int_{\widehat{\mathcal{G}^{\varepsilon}}}\frac{\chi(x)}{l^{\varepsilon}_{\mu}(x)}\,\mathrm{d}x\,, \tag{4.9}$$

which is finite for all $\varepsilon > 0$ because $\widehat{\mathcal{G}^{\varepsilon}}$ is compact and χ is supported away from 0. We define a *renormalized* resonance product by

$$Y^{\varepsilon}_{\mu} \bullet \xi^{\varepsilon} := Y^{\varepsilon}_{\mu}\circ\xi^{\varepsilon} - c^{\varepsilon}_{\mu}\,.$$

Remark 4.2.1. *Since $l^{\varepsilon}_{\mu} \approx |\cdot|^2$ (Lemma 3.3.5 together with the easy estimate $l^{\varepsilon}_{\mu} \lesssim |\cdot|^2$) we have $c^{\varepsilon}_{\mu} \approx -\log\varepsilon$ in dimension 2.*

Using Lemma 3.4.1 we can derive the following bounds.

Lemma 4.2.2. *Let ξ^{ε}, Y^{ε} and $Y^{\varepsilon}_{\mu}\bullet\xi^{\varepsilon}$ be defined on $\mathcal{G}^{\varepsilon}$ as above with $p_{\xi} \geqslant 4$ (where p^{ξ} is as in Section 3.4). For $\mu \in \boldsymbol{\mu}(\boldsymbol{\omega})$, $\zeta < 2 - d/2 - d/p_{\xi}$ and $\kappa > d/p_{\xi}$ we have*

$$\mathbb{E}\left[\,\|\xi^{\varepsilon}\|^{p_{\xi}}_{\mathcal{C}^{\zeta-2}(\mathcal{G}^{\varepsilon},\langle x\rangle^{-\kappa})}\right] + \mathbb{E}\left[\,\|Y^{\varepsilon}_{\mu}\|^{p_{\xi}}_{\mathcal{C}^{\zeta}(\mathcal{G}^{\varepsilon},\langle x\rangle^{-\kappa})}\right] + \mathbb{E}\left[\|Y^{\varepsilon}_{\mu}\bullet\xi^{\varepsilon}\|^{p_{\xi}/2}_{\mathcal{C}^{2\zeta-2}(\mathbb{R}^d,\langle x\rangle^{-2\kappa})}\right] \lesssim 1\,. \tag{4.10}$$

The involved constant is independent of ε.

Proof. Let us bound the regularity of Y^{ε}_{μ}. Recall that by Lemma 3.1.9 we have the continuous embedding (with norm uniformly bounded in ε) $\mathcal{B}^{\zeta+d/p_{\xi}}_{p_{\xi},p_{\xi}}(\mathcal{G}^{\varepsilon},\langle x\rangle^{-\kappa}) \subseteq \mathcal{C}^{\zeta}(\mathcal{G}^{\varepsilon},\langle x\rangle^{-\kappa})$. To show (4.10) it is therefore sufficient to bound for $\beta < 2 - d/2$

$$\mathbb{E}\left[\|Y^{\varepsilon}_{\mu}\|^{p_{\xi}}_{\mathcal{B}^{\beta}_{p_{\xi},p_{\xi}}(\mathcal{G}^{\varepsilon},\langle x\rangle^{-\kappa})}\right] = \sum_{-1\leqslant j\leqslant j_{\mathcal{G}^{\varepsilon}}} 2^{jp_{\xi}\beta}\sum_{z\in\mathcal{G}^{\varepsilon}}|\mathcal{G}^{\varepsilon}|\,\mathbb{E}[|\varDelta^{\mathcal{G}^{\varepsilon}}_{j}Y^{\varepsilon}_{\mu}(z)|^{p_{\varepsilon}}]\frac{1}{(1+|z|)^{\kappa p_{\xi}}}\,.$$

By assumption we have $\kappa p_\xi > d$ and can bound $\sum_{z\in\mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon|(1+|z|)^{-\kappa p_\xi} \lesssim 1$ uniformly in ε (for example by Lemma 3.5.1). It thus suffices to derive a bound for $\mathbb{E}[|\Delta_j^{\mathcal{G}^\varepsilon} Y_\mu^\varepsilon(x)|^{p_\varepsilon}]$, uniform in ε and x. Note that by (3.6)

$$\Delta_j^{\mathcal{G}^\varepsilon} Y_\mu^\varepsilon(x) = \sum_{z\in\mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon|\, \mathscr{K}_j^\varepsilon(x-z)\xi^\varepsilon(z)$$

with $\mathscr{K}_j^\varepsilon = \mathcal{F}_{\mathcal{G}^\varepsilon}^{-1}\varphi_j^{\mathcal{G}^\varepsilon}\chi/l_\mu^\varepsilon$ so that Lemma 3.4.1, Parseval's identity (3.5) and $l_\mu^\varepsilon \gtrsim |\cdot|^2$ on $\widehat{\mathcal{G}^\varepsilon}$ (from Lemma 3.3.5) imply

$$\mathbb{E}[|\Delta_j^{\mathcal{G}^\varepsilon} Y_\mu^\varepsilon(x)|^{p_\xi}] \lesssim \|\mathscr{K}_j^\varepsilon\|_{L^2(\mathcal{G}^\varepsilon)}^{p_\xi} \lesssim 2^{jp_\xi(d/2-2)}\,,$$

which proves the bound for Y_μ^ε. The bound for ξ^ε follows from the same arguments or with Lemma 3.3.4.

Now let us turn to $Y_\mu^\varepsilon \bullet \xi^\varepsilon$. A short computation shows that

$$\mathbb{E}[(Y_\mu^\varepsilon \circ \xi^\varepsilon)(x)] = \mathbb{E}[(Y_\mu^\varepsilon \cdot \xi^\varepsilon)(x)] = c_\mu^\varepsilon, \qquad x\in\mathcal{G}^\varepsilon\,,$$

and, by a similar argument as above, it now suffices to bound $Y_\mu^\varepsilon \bullet \xi^\varepsilon$ in $\mathcal{B}^\beta_{p_\xi/2,p_\xi/2}(\mathbb{R}^d,\langle x\rangle^{-2\kappa})$ for $\beta < 2-d$. We are therefore left with the task of bounding the $p_\xi/2$-th moment of $\sum_{|i-j|\leqslant 1} \Delta_i^{\mathcal{G}^\varepsilon} Y_\mu^\varepsilon \Delta_j^{\mathcal{G}^\varepsilon}\xi^\varepsilon - \mathbb{E}[\Delta_i^{\mathcal{G}^\varepsilon} Y_\mu^\varepsilon \Delta_j^{\mathcal{G}^\varepsilon}\xi^\varepsilon]$. But

$$\begin{aligned}
&\Delta_i^{\mathcal{G}^\varepsilon} Y_\mu^\varepsilon(x) \Delta_j^{\mathcal{G}^\varepsilon}\xi^\varepsilon(x) - \mathbb{E}[\Delta_i^{\mathcal{G}^\varepsilon} Y_\mu^\varepsilon(x) \Delta_j^{\mathcal{G}^\varepsilon}\xi^\varepsilon(x)]\\
&\qquad = \sum_{z_1,z_2} |\mathcal{G}^\varepsilon|^2\, \mathscr{K}_i^\varepsilon(x-z_1)\Psi^j(x-z_2)\,(\xi^\varepsilon(z_1)\xi^\varepsilon(z_2) - \mathbb{E}[\xi^\varepsilon(z_1)\xi^\varepsilon(z_2)])\\
&\qquad = \sum_{z_1,z_2} |\mathcal{G}^\varepsilon|^2\, \mathscr{K}_i^\varepsilon(x-z_1)\Psi^j(x-z_2)\xi^\varepsilon(z_1)\diamond\xi^\varepsilon(z_2)\,,
\end{aligned}$$

so that Lemma 3.4.1 yields

$$\begin{aligned}
\mathbb{E}\Big[\big|\Delta_i^{\mathcal{G}^\varepsilon} Y_\mu^\varepsilon \Delta_j^{\mathcal{G}^\varepsilon}\xi^\varepsilon - \mathbb{E}[\Delta_i^{\mathcal{G}^\varepsilon} Y_\mu^\varepsilon \Delta_j^{\mathcal{G}^\varepsilon}\xi^\varepsilon]\big|^{p_\xi/2}\Big] &\lesssim \|\mathscr{K}_i^\varepsilon\|_{L^2(\mathcal{G}^\varepsilon)}^{p_\xi/2}\|\Psi^j\|_{L^2(\mathcal{G}^\varepsilon)}^{p_\xi/2}\\
&\lesssim 2^{i(d/2-2)p_\xi/2}2^{jd/2\cdot p_\xi/2} \simeq 2^{j(d-2)\cdot p_\xi/2}\,,
\end{aligned}$$

where we used Parseval's identity, $l_\mu^\varepsilon \gtrsim |\cdot|^2$ on $\widehat{\mathcal{G}^\varepsilon}$, and that $|i-j|\leqslant 1$. □

By the compact embedding result in Lemma 2.1.26 we see that the sequences $(\mathscr{E}^\varepsilon\xi^\varepsilon)$, $(\mathscr{E}^\varepsilon Y_\mu^\varepsilon)$, and $(\mathscr{E}^\varepsilon(Y_\mu^\varepsilon\bullet\xi^\varepsilon))$ have convergent subsequences in distribution. We will see in Lemma 4.2.3 below that $\mathscr{E}^\varepsilon\xi^\varepsilon$ converges to the white noise ξ on $\mathbb{R}^2$. Consequently, the solution Y_μ^ε to $-L_\mu^\varepsilon Y_\mu^\varepsilon = \chi(D_{\mathcal{G}^\varepsilon})\xi^\varepsilon$ should, if all goes well, approach the solution of $-L_\mu Y_\mu = \chi(D_{\mathbb{R}^d})\xi := \mathcal{F}_{\mathbb{R}^d}^{-1}\big(\chi\,\mathcal{F}_{\mathbb{R}^d}\xi\big)$, i.e.

$$Y_\mu = \frac{1}{(2\pi)^2\|D_{\mathbb{R}^2}\|_\mu^2}\chi(D_{\mathbb{R}^2})\xi = \mathscr{K}_\mu^0 * \xi, \qquad \mathscr{K}_\mu^0 := \mathcal{F}_{\mathbb{R}^d}^{-1}\frac{\chi}{(2\pi)^2\|\cdot\|_\mu^2}\,. \tag{4.11}$$

where $\|\cdot\|_\mu$ is defined as in Definition 3.3.1. The limit of $\mathscr{E}^\varepsilon(Y_\mu^\varepsilon \bullet \xi^\varepsilon)$ will turn out to be the distribution

$$Y_\mu \bullet \xi(\varphi) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^2} \mathscr{K}_\mu^0(z_1 - z_2)\varphi(z_1)\xi(\mathrm{d}z_1) \diamond \xi(\mathrm{d}z_2) - (Y_\mu \prec \xi + \xi \prec Y_\mu)(\varphi) \quad (4.12)$$

for $\varphi \in \mathcal{S}_\omega(\mathbb{R}^d)$, where the first term on the right hand side denotes as in Section 3.4 the second order Wiener-Itô (or Skorohod) integral with respect to the Gaussian stochastic measure $\xi(\mathrm{d}z)$ induced by the white noise ξ. Note that $Y_\mu \bullet \xi$ is not a continuous functional of ξ, so the last convergence is not a trivial consequence of the convergence for $\mathscr{E}^\varepsilon \xi^\varepsilon$. To identify the limit of $\mathscr{E}^\varepsilon(Y_\mu^\varepsilon \bullet \xi^\varepsilon)$ we could use a diagonal sequence argument that first approximates the bilinear functional by a continuous bilinear functional as in [MW17b, HS15, CGP17]. However, having already established the machinery in Section 3.4 we can apply Lemma 3.4.2 instead.

Lemma 4.2.3. *In the setup of Lemma 4.2.2 with ξ, Y_μ and $Y_\mu \bullet \xi$ defined as above and with ζ, κ as in Lemma 4.2.2 we have for $d < 4$*

$$(\mathscr{E}^\varepsilon \xi^\varepsilon, \mathscr{E}^\varepsilon Y_\mu^\varepsilon, \mathscr{E}^\varepsilon(Y_\mu^\varepsilon \bullet \xi^\varepsilon)) \xrightarrow{\varepsilon \to 0} (\xi, Y_\mu, Y_\mu \bullet \xi)$$

in distribution in $\mathcal{C}^{\zeta-2}(\mathbb{R}^d, \langle x\rangle^{-\kappa}) \times \mathcal{C}^{\zeta}(\mathbb{R}^d, \langle x\rangle^{-\kappa}) \times \mathcal{C}^{2\zeta-2}(\mathbb{R}^d, \langle x\rangle^{-2\kappa})$.

Proof. Recall that the extension operator $\mathscr{E}^\varepsilon$ is constructed from $\psi^\varepsilon = \psi(\varepsilon\cdot)$ where the smear function ψ is symmetric and satisfies in particular $\psi \in C^\infty_{\omega,c}(\mathbb{R}^d)$ and $\psi = 1$ on some ball around 0.

Since from Lemma 4.2.2 we already know that the sequence $(\mathscr{E}^\varepsilon \xi^\varepsilon, \mathscr{E}^\varepsilon Y_\mu^\varepsilon, \mathscr{E}^\varepsilon(Y_\mu^\varepsilon \bullet \xi^\varepsilon))$ is tight in $\mathcal{C}^{\zeta-2}(\mathbb{R}^d, \langle x\rangle^{-\kappa}) \times \mathcal{C}^{\zeta}(\mathbb{R}^d, \langle x\rangle^{-\kappa}) \times \mathcal{C}^{2\zeta-2}(\mathbb{R}^d, \langle x\rangle^{-2\kappa})$, it suffices to prove the convergence after testing against $\varphi \in \mathcal{S}_\omega(\mathbb{R}^d)$:

$$(\mathscr{E}^\varepsilon \xi^\varepsilon(\varphi), \mathscr{E}^\varepsilon Y_\mu^\varepsilon(\varphi), \mathscr{E}^\varepsilon(Y_\mu^\varepsilon \bullet \xi^\varepsilon)(\varphi)) \xrightarrow{\mathrm{d}} (\xi(\varphi), Y_\mu(\varphi), Y_\mu \bullet \xi(\varphi)) \,. \quad (4.13)$$

We can even restrict ourselves to those $\varphi \in \mathcal{S}_\omega(\mathbb{R}^d)$ with $\mathcal{F}_{\mathbb{R}^d}\varphi \in C^\infty_{\omega,c}(\mathbb{R}^d)$, which implies $\operatorname{supp} \mathcal{F}_{\mathbb{R}^d}\varphi \subseteq \widehat{\mathcal{G}^\varepsilon}$ and $\mathcal{F}^{-1}_{\mathbb{R}^d}(\psi^\varepsilon \mathcal{F}_{\mathbb{R}^d}\varphi) = \varphi$ for ε small enough, which we will assume from now on. Note that $\operatorname{supp} \mathcal{F}_{\mathbb{R}^d}\varphi \subseteq \widehat{\mathcal{G}^\varepsilon}$ implies

$$\mathcal{F}_{\mathcal{G}^\varepsilon}\varphi = \mathcal{F}_{\mathbb{R}^d}\varphi|_{\widehat{\mathcal{G}^\varepsilon}} \quad (4.14)$$

since by definition of $\mathcal{F}^{-1}_{\mathcal{G}^\varepsilon}$

$$\mathcal{F}^{-1}_{\mathcal{G}^\varepsilon}\mathcal{F}_{\mathbb{R}^d}\varphi = (\mathcal{F}^{-1}_{\mathbb{R}^d}\mathcal{F}_{\mathbb{R}^d}\varphi)|_{\mathcal{G}^\varepsilon} = \varphi|_{\mathcal{G}^\varepsilon} \,.$$

Let us first show the convergence of (4.13) in every component.

To show the convergence of $\mathscr{E}^\varepsilon \xi^\varepsilon$ to white noise note that we have from (3.21) the following formula

$$\begin{aligned}\mathscr{E}^\varepsilon \xi^\varepsilon(\varphi) &= \sum_{z\in\mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon|\,(\mathcal{F}^{-1}_{\mathbb{R}^d}\psi^\varepsilon * \varphi)(z)\xi^\varepsilon(z) = \sum_{z\in\mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon|\,\mathcal{F}^{-1}_{\mathbb{R}^d}(\psi^\varepsilon\,\mathcal{F}_{\mathbb{R}^d}\varphi)(z)\xi^\varepsilon(z)\\ &= \sum_{z\in\mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon|\varphi(z)\xi^\varepsilon(z)\end{aligned}$$

where we used in the first step that ψ^ε is symmetric and in the last step that $\mathcal{F}^{-1}_{\mathbb{R}^d}(\psi^\varepsilon \mathcal{F}_{\mathbb{R}^d}\varphi) = \varphi$ by our choice of φ and ε. Using Lemma 3.4.2 and relation (4.14) the convergence of $\mathscr{E}^\varepsilon \xi^\varepsilon(\varphi)$ to $\xi(\varphi)$ follows.

For the limit of $\mathscr{E}^\varepsilon Y^\varepsilon$ we use the following formula, which is derived by the same argument as above:

$$\mathscr{E}^\varepsilon Y^\varepsilon_\mu(\varphi) = \sum_{z_1,\, z_2\in\mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon|^2\,\varphi(z_1)\mathscr{K}^\varepsilon_\mu(z_2 - z_1)\xi^\varepsilon(z_2)$$

with $\mathscr{K}^\varepsilon_\mu = \mathcal{F}^{-1}_{\mathcal{G}^\varepsilon}\frac{\chi}{l^\varepsilon_\mu}$. In view of Lemma 3.4.2 it then suffices to note that $\hat{f}^\varepsilon := \mathcal{F}_{\mathcal{G}^\varepsilon}(\varphi *_{\mathcal{G}^\varepsilon} \mathscr{K}^\varepsilon_\mu) = \mathcal{F}_{\mathcal{G}^\varepsilon}\varphi\cdot\frac{\chi}{l^\varepsilon_\mu} \overset{(4.14)}{=} \mathcal{F}_{\mathbb{R}^d}\varphi\cdot\frac{\chi}{l^\varepsilon_\mu}$ is due to Lemma 3.3.5 dominated by $\chi/|\cdot|^2$ on $\widehat{\mathcal{G}^\varepsilon}$ and converges to

$$\mathcal{F}_{\mathbb{R}^d}\varphi\,\chi/\big((2\pi)^2\|\cdot\|^2_\mu)\big)$$

by the explicit formula for l^ε_μ in (3.32).

We are left with the convergence of the third component. Since $\mathscr{E}^\varepsilon\xi^\varepsilon \to \xi$ and $\mathscr{E}^\varepsilon Y^\varepsilon_\mu \to Y_\mu$ we obtain via the ($\mathscr{E}$)-Property of the paraproduct

$$\lim_{\varepsilon\to 0}\mathscr{E}^\varepsilon(Y^\varepsilon_\mu \prec \xi^\varepsilon) = \lim_{\varepsilon\to 0}\mathscr{E}^\varepsilon Y^\varepsilon_\mu \prec \mathscr{E}^\varepsilon\xi^\varepsilon = Y_\mu \prec \xi$$

and similarly one gets $\mathscr{E}^\varepsilon(\xi^\varepsilon \prec Y^\varepsilon_\mu) \to \xi \prec Y_\mu$. We can therefore show instead

$$\mathscr{E}^\varepsilon\left(Y^\varepsilon_\mu\xi^\varepsilon - \mathbb{E}[Y^\varepsilon_\mu\xi^\varepsilon]\right)(\varphi) \to (Y_\mu \bullet \xi + \xi \prec Y_\mu + Y_\mu \prec \xi)(\varphi)\,. \tag{4.15}$$

Note that we have the representations

$$\begin{aligned}\mathscr{E}^\varepsilon\left(Y^\varepsilon_\mu\xi^\varepsilon - \mathbb{E}[Y^\varepsilon_\mu\xi^\varepsilon]\right)(\varphi) &= \sum_{z_1, z_2\in\mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon|^2\varphi(z_1)\mathscr{K}^\varepsilon_\mu(z_1 - z_2)\,\xi^\varepsilon(z_1)\diamond\xi^\varepsilon(z_2)\,,\\ (Y_\mu \bullet \xi + \xi \prec Y_\mu + Y_\mu \prec \xi)(\varphi) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\varphi(z_1)\mathscr{K}^0_\mu(z_1 - z_2)\,\xi(\mathrm{d}z_1)\diamond\xi(\mathrm{d}z_2)\end{aligned}$$

with $\mathscr{K}^\varepsilon_\mu$ as above and $\mathscr{K}^0_\mu$ as in (4.11). The $(\mathcal{G}^\varepsilon)^2$-Fourier transform of $\varphi(z_1)\mathscr{K}^\varepsilon_\mu(z_1 - z_2)$ is $\hat{\varphi}_{\mathrm{ext}}(x_1 - x_2)\chi(x_2)/l^\varepsilon_\mu(x_2)$ for $x_1, x_2 \in \widehat{\mathcal{G}^\varepsilon}$, where $\hat{\varphi}_{\mathrm{ext}}$ denotes the periodic extension from (3.10) for $\mathcal{F}_{\mathbb{R}^d}\varphi|_{\widehat{\mathcal{G}^\varepsilon}} \in C^\infty_{\omega,c}(\widehat{\mathcal{G}^\varepsilon})$ (recall again that $\operatorname{supp}\mathcal{F}_{\mathbb{R}^d}\varphi \subseteq \widehat{\mathcal{G}^\varepsilon}$). We can therefore apply Lemma 3.4.2 since for $d < 4$ the function $(\chi(x_2)/l^\varepsilon(x_2))^2 \lesssim \mathbf{1}_{|x|\gtrsim 1}/|x|^4$ is integrable on $\widehat{\mathcal{G}^\varepsilon}$ and thus we obtain (4.15).

We have shown the convergence in distribution of all the components in (4.13). By Lemma 3.4.2 we can take any linear combination of these components and still get the convergence from the same estimates, so that (4.13) follows from the Cramér-Wold Theorem. $\square$

4.3 Weak Universality

We are now ready to prove the result announced at the beginning of this chapter. Consider thus a lattice sequence $\mathcal{G}^\varepsilon$ as in Definition 3.1.2 in dimension $d = 2$. The family $\big(\xi^\varepsilon(z)\big)_{z\in\mathcal{G}^\varepsilon}$ is an approximation to white noise on $\mathcal{G}^\varepsilon$ as in Section 3.4 with moments $p_\xi > 14$, compare Remark 4.3.1 below. Finally u^ε is the solution to the equation (4.2):

$$\mathscr{L}^\varepsilon_\mu u^\varepsilon = F^\varepsilon(u^\varepsilon)(\xi^\varepsilon - F'(0)c^\varepsilon_\mu), \qquad u^\varepsilon(0) = |\mathcal{G}^\varepsilon|^{-1}\mathbf{1}_{\cdot=0}\,,$$

where $c^\varepsilon_\mu \approx |\log\varepsilon|$ is as in Section 4.2 build from ξ^ε and some function $Y^\varepsilon = \chi(D_{\mathcal{G}^\varepsilon})\xi^\varepsilon \in \mathcal{S}'(\mathcal{G}^\varepsilon)$ and given by identity (4.9). Let us recall that $F \in C^2(\mathbb{R};\mathbb{R})$ is assumed to have a bounded second derivative and to satisfy $F(0) = 0$. We used above the notation $F^\varepsilon := \varepsilon^{-2}F(\varepsilon^2\cdot)$.

In this section we want to study the convergence of the extensions $\mathscr{E}^\varepsilon u^\varepsilon$. The key statement will be the a priori estimate in Lemma 4.3.3. The convergence of $\mathscr{E}^\varepsilon u^\varepsilon$ to the continuous solution on $\mathbb{R}^2$, constructed in Corollary 4.3.5, will be proven in Theorem 4.3.6. We first fix the relevant parameters.

Preliminaries

Throughout this Section we use the same $p \in [1,\infty]$, $\sigma \in (0,1)$, a polynomial weight $\langle x\rangle^{-\kappa}$ for some $\kappa > 2/p_\xi > 1/7$ and a time dependent sub-exponential weight $(e^\sigma_{l+t})_{t\in[0,T]}$. We further fix an arbitrarily large time horizon $T > 0$ and require $l \leqslant -T$ for the parameter in the weight e^σ_l. Then we have $1 \leqslant e^\sigma_{l+t} \leqslant (e^\sigma_{l+t})^2$ for any $t \leqslant T$, which will be used to control a quadratic term that comes from the Taylor expansion of the non-linearity F^ε.

In this subsection we fix a parameter

$$\alpha \in (2/3 - 2/3\cdot\kappa/\sigma, 1 - 2/p_\xi - 2\kappa/\sigma) \tag{4.16}$$

with $\kappa/\sigma \in (2/p_\xi, 1)$ small enough such that the interval in is non-empty, which is possible since $2/p_\xi < 1/7$.

Remark 4.3.1 (Why 14+ moments)**.** *Let us sketch where the boundaries of the interval* (4.16) *come from. The parameter α will measure the regularity of u^ε below. The upper boundary, that is $1 - 2/p_\xi - 2\kappa/\sigma$, arises due to the fact that we cannot expect u^ε to be better than Y^ε, which has regularity below $1 - 2/p_\xi$ due to Lemma 4.2.2. The correction $-2\kappa/\sigma$ is just the price one pays in the Schauder estimate in Lemma 4.1.2 for the "weight change". The lower bound $2/3 - 2/3 \cdot \kappa/\sigma$ is a criterion for our paracontrolled approach below to work. We increase below the regularity α of our solutions, by subtraction of a paraproduct, to 2α. By Lemma 3.2.2 this allows to control uniformly products with ξ^ε provided*

$$2\alpha + (\alpha + 2\kappa/\sigma - 2) > 0\,,$$

where we expressed the regularity of ξ^ε via $\alpha + 2\kappa/\sigma - 2$. This condition can be reshaped to $\alpha > 2/3 - 2/3 \cdot \kappa/\sigma$, explaining the lower bound.

The interval (4.16) *can only be non-empty if*

$$2/3 - 2/3 \cdot \kappa/\sigma < 1 - 2/p_\xi - 2\kappa/\sigma \;\Leftrightarrow\; 2/3 < 1 - 2/p_\xi - 4/3 \cdot \kappa/\sigma$$

Lemma 4.2.2 forces us to take $\kappa/\sigma > 2/p_\xi$ so that the the right hand side can only be true provided $2/3 < 1 - 2/p_\xi - 4/3 \cdot 2/p_\xi$ which is equivalent to

$$p_\xi > 14\,.$$

Let us mention the simple facts $2\alpha + 2\kappa/\sigma, 2\alpha + 4\kappa/\sigma \in (0,2)$, $\alpha + \kappa/\sigma, \alpha + 2\kappa/\sigma \in (0,1)$ and $3\alpha + 2\kappa/\sigma - 2 > 0$ that follow from (4.16) and which we will use frequently below.

We will assume that the initial conditions u_0^ε are uniformly bounded in $\mathcal{C}_p^0(\mathcal{G}^\varepsilon, e_l^\sigma)$ and are such that $\mathscr{E}^\varepsilon u_0^\varepsilon$ converges in $\mathcal{S}'_\omega(\mathbb{R}^2)$ to some u_0. For $u_0^\varepsilon = |\mathcal{G}^\varepsilon|^{-1}\mathbf{1}_{\cdot=0}$ it is easily verified that this is indeed the case and the limit is the Dirac delta, $u_0 = \delta$.

Recall that we aim at showing that (the extension of) the solution u^ε to

$$\mathscr{L}_\mu^\varepsilon u^\varepsilon = F(u^\varepsilon)(\xi^\varepsilon - c_\mu^\varepsilon), \qquad u^\varepsilon(0) = u_0^\varepsilon = |\mathcal{G}^\varepsilon|^{-1}\mathbf{1}_{\cdot=0} \tag{4.17}$$

converges to the solution of

$$\mathscr{L}_\mu u = F'(0) u \blacklozenge \xi, \qquad u(0) = u_0 = \delta\,, \tag{4.18}$$

where $u \blacklozenge \xi$ is a suitably renormalized product defined in Corollary 4.3.5 below. Our solutions will be objects in the parabolic space $\mathscr{L}_{p,T}^{\alpha;\alpha}$ which does not require continuity at $t = 0$. A priori there is thus no obvious meaning for the Cauchy problems (4.17), (4.18) (although of course for (4.17) we could use the pointwise interpretation). We follow the common interpretation for distributions $u^\varepsilon, u \in \left(C_{\omega,c}^\infty\right)'(\mathbb{R}^{1+2})$

(compare for example [Tri92, Definition 3.3.4]) to require $\operatorname{supp} u^\varepsilon$, $\operatorname{supp} u \subseteq \mathbb{R}_+ \times \mathbb{R}^2$ and

$$\begin{aligned}\mathscr{L}_\mu^\varepsilon u^\varepsilon &= F(u^\varepsilon)(\xi^\varepsilon - c_\mu^\varepsilon) + \delta \otimes u_0^\varepsilon\,,\\ \mathscr{L}_\mu u &= F'(0) u \blacklozenge \xi + \delta \otimes u_0\,,\end{aligned}$$

in the distributional sense on $(-\infty, T)$, where $\otimes$ denotes the tensor product between distributions. Since we mostly work with the mild formulation of these equations the distributional interpretation will not play a crucial role. Some care is needed to check that the only distributional solutions are mild solutions, since the distributional Cauchy problem for the heat equation is not uniquely solvable [Tyc35]. However, under generous growth conditions for u, u^ε for $x \to \infty$ (compare [Fri64]) there is a unique solution. In our case this fact can be checked by considering the Fourier transform of u, u^ε in space.

A priori estimates

We will work with the following space of paracontrolled distributions.

Definition 4.3.2 (Paracontrolled distribution for 2d PAM)**.** *We identify a pair*

$$(u^{\varepsilon,Y}, u^{\varepsilon,\sharp})\colon [0,T] \to \mathcal{S}'_\omega(\mathcal{G}^\varepsilon)^2$$

with $u^\varepsilon \in \mathcal{S}'_\omega(\mathcal{G}^\varepsilon)$ via $u^\varepsilon = u^{\varepsilon,Y} \lll Y^\varepsilon + u^{\varepsilon,\sharp}$ and introduce a norm

$$\|u^\varepsilon\|_{\mathscr{D}^{\nu,\delta}_{p,T}} := \|(u^{\varepsilon,Y}, u^{\varepsilon,\sharp})\|_{\mathscr{D}^{\nu,\delta}_{p,T}} := \|u^{\varepsilon,Y}\|_{\mathscr{L}^{\nu/2,\delta}_{p,T}(\mathcal{G}^\varepsilon, e_l^\sigma)} + \|u^{\varepsilon,\sharp}\|_{\mathscr{L}^{\nu,\delta+\alpha}_{p,T}(\mathcal{G}^\varepsilon, e_l^\sigma)} \tag{4.19}$$

for α as above and $\nu \geqslant 0$, $\delta > 0$. We denote the corresponding space by $\mathscr{D}^{\nu,\delta}(\mathcal{G}^\varepsilon, e_l^\sigma)$. If the norm (4.19) is bounded for a sequence $u^\varepsilon = u^{\varepsilon,Y} \lll Y^\varepsilon + u^{\varepsilon,\sharp}$ we say that u^ε is paracontrolled by Y^ε.

The notation $\mathscr{D}^{\nu,\delta}_{p,T}$ is chosen on purpose close to the notation $\mathcal{D}^\gamma$ we used in Section 2.3 for the space of modelled distribution. In fact, by the connections we unravel in Section 5.1 and 5.2 below, one should have (for $\varepsilon \to 0$) a correspondence

$$\mathscr{D}^{0,\delta}_{\infty,T} \text{ “}\simeq\text{” } \mathcal{D}^{\alpha+\delta}([0,T] \times \mathbb{R}^2; \mathcal{T})\,,$$

for a suitable graded vector-space $\mathcal{T}$, in mind. However, this is not quite true as we use, for example, space-time paraproducts instead of modified paraproducts in Chapter 5 and some care is needed when working on compact time intervals (Section 5.3).

Let us fix a common bound on the data: We define (compared to Lemma 4.2.2 we have $\zeta = \alpha + 2\kappa/\sigma$)

$$M_\varepsilon := 1 + \|\xi^\varepsilon\|_{\mathcal{C}^{\alpha+2\kappa/\sigma-2}(\mathcal{G}^\varepsilon, \langle x\rangle^{-\kappa})} + \|Y^\varepsilon\|_{\mathcal{C}^{\alpha+2\kappa/\sigma}(\mathcal{G}^\varepsilon, \langle x\rangle^{-\kappa})} + \|Y^\varepsilon \bullet \xi^\varepsilon\|_{\mathcal{C}^{2\alpha+4\kappa/\sigma-2}(\mathcal{G}^\varepsilon, \langle x\rangle^{-2\kappa})}\,. \tag{4.20}$$

The following a priori estimates will allow us to set up a Picard iteration below.

Lemma 4.3.3 (A priori estimates). *In the setup above and for $\nu \in \{0, \alpha\}$ define a map*

$$u^\varepsilon = u^{\varepsilon,Y} \lll Y^\varepsilon + u^{\varepsilon,\sharp} \quad \longmapsto (v^{\varepsilon,Y}, v^{\varepsilon,\sharp}) \tag{4.21}$$

for $u^\varepsilon = u^{\varepsilon,Y} \lll Y^\varepsilon + u^{\varepsilon,\sharp} \in \mathscr{D}^{\nu,\alpha}_{p,T}$ and $u_0^\varepsilon \in \mathcal{S}'_\omega(\mathcal{G}^\varepsilon)$ via

$$\mathscr{L}^\varepsilon_\mu v^\varepsilon := F^\varepsilon(u^\varepsilon)\xi^\varepsilon - F^\varepsilon(u^{\varepsilon,Y}/F'(0))F'(0)c^\varepsilon_\mu, \qquad v^\varepsilon(0) = u_0^\varepsilon\,, \tag{4.22}$$

$$v^{\varepsilon,\sharp} := v^\varepsilon - F'(0)u^\varepsilon \lll Y^\varepsilon\,. \tag{4.23}$$

and $v^{\varepsilon,Y} := F'(0)v^\varepsilon$. We then have for $\nu \in \{0, \alpha\}$ the bound

$$\begin{aligned}\|(v^{\varepsilon,Y}, v^{\varepsilon,\sharp})\|_{\mathscr{D}^{\nu,\alpha}_{p,T}} \lesssim_{M_\varepsilon} &\mathbf{1}_{\nu=0}\left(\|v^{\varepsilon,\sharp}(0)\|_{\mathcal{C}^{2\alpha}_p(\mathcal{G}^\varepsilon,e_l^\sigma)} + \|u^{\varepsilon,\sharp}(0)\|_{\mathcal{C}^{2\alpha}_p(\mathcal{G}^\varepsilon,e_l^\sigma)} + \|u^{\varepsilon,Y}(0)\|_{\mathcal{C}^{\alpha}_p(\mathcal{G}^\varepsilon,e_l^\sigma)}\right)\\ &+ \mathbf{1}_{\nu=\alpha}\left(\|v^{\varepsilon,\sharp}(0)\|_{\mathcal{C}^{0}_p(\mathcal{G}^\varepsilon,e_l^\sigma)}\right) + T^{(\alpha-\delta)/2}\left(\|u^\varepsilon\|_{\mathscr{D}^{\nu,\alpha}_{p,T}} + \varepsilon^\eta\|u^\varepsilon\|^2_{\mathscr{D}^{\nu,\alpha}_{p,T}}\right)\end{aligned}$$

for $\delta \in (2 - 2\alpha - 2\kappa/\sigma, \alpha)$, some $\eta > 0$ and with $v^{\varepsilon,\sharp}(0) = u_0^\varepsilon - F'(0)(u^\varepsilon \lll Y^\varepsilon)(0)$. The involved constant can be chosen proportional to $(1 + \|F''\|_{L^\infty(\mathbb{R})})M_\varepsilon^2$.

Remark 4.3.4. *The complicated formulation of* (4.22) *is necessary because when we expand the singular product on the right hand side we get*

$$F^\varepsilon(u^\varepsilon)\xi^\varepsilon = F'(0)(C(u^{\varepsilon,Y}, Y^\varepsilon, \xi^\varepsilon) + u^{\varepsilon,Y}(Y^\varepsilon \circ \xi^\varepsilon)) + \ldots\,,$$

so to obtain the right renormalization we need to subtract $F'(0)u^{\varepsilon,Y}c^\varepsilon_\mu$, which is exactly what we get if we Taylor expand the second addend on the right hand side of (4.22)*. Of course, if u is a fixed point of the map defined in* (4.22)*,* (4.23)*, then $u^{\varepsilon,Y} = F'(0)u^\varepsilon$ and the "renormalization term" is just $F^\varepsilon(u^\varepsilon)F'(0)c^\varepsilon_\mu$.*

Proof. We assume $M_\varepsilon \leqslant 1$ for simplicity, the quadratic dependence on M_ε of the derived bound will be clear from the proof below. The solution to (4.22), (4.23) can be constructed using the Green's function $\mathcal{F}^{-1}_{\mathcal{G}^\varepsilon}e^{-t\ell^\varepsilon}$ and Duhamel's principle. We derive the bounds similar in spirit to [GP15b]. To uncluster the notation a bit, we will drop the upper index ε, and the lower index μ, on u, v, Y, $\mathscr{L}$, c, ... in this proof. We show both estimates at once by denoting by ν either 0 or α.

Throughout the proof we will use the fact that

$$\|u\|_{\mathscr{L}^{\nu/2,\alpha}_{p,T}(\mathcal{G}^\varepsilon,e_l^\sigma)} = \|u^Y \lll Y + u^\sharp\|_{\mathscr{L}^{\nu/2,\alpha}_{p,T}(\mathcal{G}^\varepsilon,e_l^\sigma)} \lesssim \|u\|_{\mathscr{D}^{\nu,\beta}_{p,T}} \tag{4.24}$$

for $\beta \in (0, \alpha]$ which follows from Lemma 4.1.6. We first estimate

$$\|v\|_{\mathscr{L}^{\nu/2,\alpha}_{p,T}(\mathcal{G}^\varepsilon,e_l^\sigma)} = \|F'(0)u \lll Y + v^\sharp\|_{\mathscr{L}^{\nu/2,\alpha}_{p,T}(\mathcal{G}^\varepsilon,e_l^\sigma)} \overset{(4.24)}{\lesssim} \|u\|_{\mathscr{D}^{\nu,\delta}_{p,T}} + \|v^\sharp\|_{\mathscr{L}^{\nu,2\alpha}_{p,T}(\mathcal{G}^\varepsilon,e_l^\sigma)}\,, \tag{4.25}$$

where we used Lemma 4.1.5 and Lemma 4.1.3 in the second step. Next, let us bound $\|v^\sharp\|_{\mathscr{L}^{\nu,2\alpha}_{p,T}(\mathcal{G}^\varepsilon,e^\sigma_l)}$. To this end we split via a Taylor expansion for $F^\varepsilon(u)$ and $F^\varepsilon(u^Y/F'(0))$

$$
\begin{aligned}
\mathscr{L}v^\sharp &= \mathscr{L}(v - F'(0)u \lll Y)\\
&= F^\varepsilon(u)\xi - F^\varepsilon(u^Y/F'(0))F'(0)c - F'(0)\mathscr{L}(u \lll Y)\\
&= F'(0)u\xi - F^\varepsilon(u^Y/F'(0))F'(0)c - F'(0)\mathscr{L}(u \lll Y) + R(u)u^2\xi\\
&= F'(0)[u \prec (\xi-\bar{\xi}) + u \prec \bar{\xi} - u \lll \bar{\xi} + u \lll \bar{\xi} - \mathscr{L}(u \lll Y) + \xi \prec u && (\prec)\\
&\qquad + C(u^Y, Y, \xi) + u^Y(Y \bullet \xi) && (\circ)\\
&\qquad + u^\sharp \circ \xi] && (\sharp)\\
&\quad + R(u)\cdot u^2\xi && (R_u)\\
&\quad - R(u^Y)\cdot (u^Y)^2 c/F'(0)\,, && (R_{u^Y})
\end{aligned}
$$

where $\bar{\xi} = \chi(D_{\mathcal{G}^\varepsilon})\xi$ so that $\mathscr{L}Y = \bar{\xi}$ and $\xi - \bar{\xi} \in \bigcap_{\beta\in\mathbb{R}} \mathcal{C}^\beta_\infty(\mathcal{G}^\varepsilon, \langle x\rangle^{-\kappa})$ and where $R(u) = \varepsilon^2 \int_0^1 F''(\lambda\varepsilon^2 u)\mathrm{d}\lambda$. We have by Lemmas 3.2.2, 4.1.7 the inequality $\|(\prec)\|_{\mathcal{M}^\nu_T\mathcal{C}^{2\alpha+2\kappa/\sigma-2}_p(\mathcal{G}^\varepsilon,e^\sigma_l\langle x\rangle^{-\kappa})} \lesssim \|u\|_{\mathscr{L}^{\nu/2,\alpha}_{p,T}(\mathcal{G}^\varepsilon,e^\sigma_l)} \overset{(4.24)}{\lesssim} \|u\|_{\mathscr{D}^{\nu,\delta}_{p,T}}$ and further with Lemma 3.2.3 and Lemma 3.2.2 the estimate $\|(\circ)\|_{\mathcal{M}^\nu_T\mathcal{C}^{2\alpha+4\kappa/\sigma-2}(\mathcal{G}^\varepsilon,e^\sigma_l\langle x\rangle^{-2\kappa})} \lesssim \|u\|_{\mathscr{D}^{\nu,\delta}_{p,T}}$, while the term $(\sharp)$ can be bounded with Lemma 3.2.2 by $\|u^\sharp\circ\xi\|_{\mathcal{M}^\nu_T\mathcal{C}^{2\alpha+2\kappa/\sigma-2}_p(\mathcal{G}^\varepsilon,e^\sigma_l\langle x\rangle^{-\kappa})} \lesssim \|u^\sharp\|_{\mathscr{L}^{\nu,\alpha+\delta}_{p,T}(\mathcal{G}^\varepsilon,e^\sigma_l)} \leqslant \|u\|_{\mathscr{D}^{\nu,\delta}_{p,T}}$. To estimate (R_u) we use the simple bounds $\|\varepsilon^{\beta'}f\|_{\mathcal{C}^{\beta+\beta'}_q(\mathcal{G}^\varepsilon,\rho)} \lesssim \|f\|_{\mathcal{C}^\beta_q(\mathcal{G}^\varepsilon,\rho)}$ for $\beta\in\mathbb{R}$, $\beta'>0$, $q\in[1,\infty]$, $\rho\in\boldsymbol{\rho}(\omega)$ and $\|\varepsilon^{-\beta}f\|_{L^q(\mathcal{G}^\varepsilon,\rho)} \lesssim \varepsilon^{-\beta}\sum_{j\lesssim j_{\mathcal{G}^\varepsilon}} 2^{-j\beta}\|f\|_{\mathcal{C}^\beta_q(\mathcal{G}^\varepsilon,\rho)} \lesssim \|f\|_{\mathcal{C}^\beta_q(\mathcal{G}^\varepsilon,\rho)}$ for $\beta<0$, $q\in[1,\infty]$, $\rho\in\boldsymbol{\rho}(\omega)$ and the assumption $F''\in L^\infty(\mathbb{R})$ to obtain for $\eta'>0$, using in the first line $e^\sigma_{l+t} \leqslant (e^\sigma_{l+t})^2$,

$$
\begin{aligned}
&\|(R_u)\|_{\mathcal{M}^\nu_T\mathcal{C}^{2\alpha+2\kappa/\sigma-2}_p(\mathcal{G}^\varepsilon,e^\sigma_l\langle x\rangle^{-\kappa})}\\
&\lesssim \|F''\|_{L^\infty(\mathbb{R})}\|\varepsilon^{\alpha+2\kappa/\sigma}u^2\|_{\mathcal{M}^\nu L^p(\mathcal{G}^\varepsilon,e^\sigma_l)}\,\|\varepsilon^{2-(\alpha+2\kappa/\sigma)}\xi\|_{L^\infty(\mathcal{G}^\varepsilon,\langle x\rangle^{-\kappa})}\\
&\lesssim \|\varepsilon^{\alpha+2\kappa/\sigma}u^2\|_{\mathcal{M}^\nu_T L^p(\mathcal{G}^\varepsilon,(e^\sigma_l)^2)}\,\|\xi\|_{\mathcal{C}^{\alpha+2\kappa/\sigma-2}(\mathcal{G}^\varepsilon,\langle x\rangle^{-\kappa})}\\
&\lesssim \|\varepsilon^{\alpha/2+\kappa/\sigma}u\|^2_{\mathcal{M}^{\nu/2}_T L^{2p}(\mathcal{G}^\varepsilon,e^\sigma_l)} \lesssim \|\varepsilon^{\alpha/2+\kappa/\sigma}u\|^2_{\mathcal{M}^{\nu/2}_T\mathcal{C}^{d/2p+\eta'}_p(\mathcal{G}^\varepsilon,e^\sigma_l)}\\
&\leqslant \|\varepsilon^{\alpha/2+\kappa/\sigma}u\|^2_{\mathcal{M}^{\nu/2}_T\mathcal{C}^{1+\eta'}_p(\mathcal{G}^\varepsilon,e^\sigma_l)} \lesssim \|\varepsilon^{\alpha/2+\kappa/\sigma-(1+\eta'-\alpha)}u\|^2_{\mathcal{M}^{\nu/2}_T\mathcal{C}^\alpha_p(\mathcal{G}^\varepsilon,e^\sigma_l)}\\
&\lesssim \varepsilon^{3\alpha+2\kappa/\sigma-2(1+\eta')}\|u\|^2_{\mathscr{D}^{\nu,\delta}_{p,T}}\,,
\end{aligned}
$$

so that for sufficiently small $\eta'>0$ we can choose $\eta\in(0,3\alpha+2\kappa/\sigma-2(1+\eta')]$.

Similarly we get for (a different) $\eta' \in (0,\delta)$

$$\begin{aligned}\|(R_{u^Y})\|_{\mathcal{M}_T^{\nu}\mathcal{C}_p^{2\alpha+2\kappa/\sigma-2}(\mathcal{G}^\varepsilon, e_l^\sigma\langle x\rangle^{-\kappa})} &\lesssim \|F''\|_{L^\infty(\mathbb{R})} c \|\varepsilon u^Y\|^2_{\mathcal{M}_T^{\nu/2}L^{2p}(\mathcal{G}^\varepsilon,e_l^\sigma)} \\ &\lesssim c\|\varepsilon u^Y\|^2_{\mathcal{M}_T^{\nu/2}\mathcal{C}_p^{1+\eta'}(\mathcal{G}^\varepsilon,e_l^\sigma)} \\ &\lesssim \varepsilon^{2(\delta-\eta')}\log(\varepsilon)\|u^Y\|^2_{\mathcal{M}_T^{\nu/2}\mathcal{C}_p^{\delta}(\mathcal{G}^\varepsilon,e_l^\sigma)} \lesssim \varepsilon^{\eta}\|u\|^2_{\mathscr{D}^{\nu,\delta}_{p,T}}\,.\end{aligned}$$

where we chose $\eta \in (0, \delta - \eta')$. The Schauder estimates of Lemma 4.1.2 yield on these grounds

$$\begin{aligned}\|v^\sharp\|_{\mathscr{L}^{\nu,2\alpha}_{p,T}(\mathcal{G}^\varepsilon,e_l^\sigma)} &\lesssim \mathbf{1}_{\nu=\alpha}\|v^\sharp(0)\|_{\mathcal{C}_p^0(\mathcal{G}^\varepsilon,e_l^\sigma)} + \mathbf{1}_{\nu=0}\|v^\sharp(0)\|_{\mathcal{C}_p^{2\alpha}(\mathcal{G}^\varepsilon,e_l^\sigma)} + \|u\|_{\mathscr{D}^{\nu,\delta}_{p,T}} + \varepsilon^\eta\|u\|^2_{\mathscr{D}^{\nu,\delta}_{p,T}} \\ &\lesssim \mathbf{1}_{\nu=\alpha}\|v^\sharp(0)\|_{\mathcal{C}_p^0(\mathcal{G}^\varepsilon,e_l^\sigma)} + T^{(\alpha-\delta)/2}(\|u\|_{\mathscr{D}^{\nu,\alpha}_{p,T}} + \varepsilon^\eta\|u\|^2_{\mathscr{D}^{\nu,\alpha}_{p,T}}) \\ &\quad + \mathbf{1}_{\nu=0}\Big(\|v^\sharp(0)\|_{\mathcal{C}_p^{2\alpha}(\mathcal{G}^\varepsilon,e_l^\sigma)} + \|u^{\varepsilon,\sharp}(0)\|_{\mathcal{C}_p^{2\alpha}(\mathcal{G}^\varepsilon,e_l^\sigma)} + \|u^{\varepsilon,Y}(0)\|_{\mathcal{C}_p^{\alpha}(\mathcal{G}^\varepsilon,e_l^\sigma)}\Big)\,,\end{aligned}$$

where in the last step we used Lemma 4.1.3. In combination with (4.25) the claim follows. □

Convergence to the continuum

It is straightforward to redo our computations in the continuous case which leads to the existence of a solution to the continuous linear parabolic Anderson model on $\mathbb{R}^2$, a result which was already established in [HL15]. Since the continuous analogue of our approach is a one-to-one translation of the discrete statements and definitions above we do not provide the details.

Corollary 4.3.5. *For any $u_0 \in \mathcal{C}_p^0(\mathbb{R}^d, e_l^\sigma)$ and $\mu \in \boldsymbol{\mu}(\omega_\sigma^{\exp})$ there is a unique solution $u = F'(0)u \lll Y + u^\sharp \in \mathscr{D}^{\nu,\beta}_{p,T}(\mathbb{R}^d, e_l^\sigma)$, $\beta \in (2/3, 1)$, $\nu \in [\beta, 1)$ to*

$$\mathscr{L}_\mu u = F'(0)u \blacklozenge \xi, \qquad u(0) = u_0\,,$$

where ξ is white noise on $\mathbb{R}^2$, $\mathscr{L}_\mu$ is defined as in section 3.3 and where

$$u \blacklozenge \xi := \xi \prec u + u \prec \xi + F'(0)C(u, Y, \xi) + F'(0)u(Y \bullet \xi) + u^\sharp \circ \xi$$

with Y, $Y \bullet \xi$ as in (4.11), (4.12).

Sketch of the proof. Redoing the computations in the continuous case leads to the continuous version of the a priori estimates of Lemma 4.3.3, without the quadratic term:

$$\begin{aligned}\|(F'(0)v, v^\sharp)\|_{\mathscr{D}^{\nu,\beta}_{p,T}} &\lesssim_M \|v^\sharp(0)\|_{\mathcal{C}_p^0(\mathbb{R}^d,e_l^\sigma)} + T^{(\beta-\delta)/2}\,\|u\|_{\mathscr{D}^{\nu,\beta}_{p,T}} \\ \|(F'(0)v, v^\sharp)\|_{\mathscr{D}^{0,\beta}_{p,T}} &\lesssim_M \|v^\sharp(0)\|_{\mathcal{C}_p^{2\beta}(\mathbb{R}^d,e_l^\sigma)} + \|u^\sharp(0)\|_{\mathcal{C}_p^{2\beta}(\mathbb{R}^d,e_l^\sigma)} \\ &\quad + \|u^Y(0)\|_{\mathcal{C}_p^{\beta}(\mathbb{R}^d,e_l^\sigma)} + T^{(\beta-\delta)/2}\|u\|_{\mathscr{D}^{0,\beta}_{p,T}}\end{aligned}$$

for $v = F'(0)u \ll Y + v^\sharp$, $\mathscr{L}_\mu v = F'(0)u \blacklozenge \xi$, $v(0) = u(0) = u_0$. Choosing $T > 0$ small enough we can set up a Picard iteration (e.g. starting in $t \mapsto e^{t\mathscr{L}} u_0 =: 0 \ll Y + u^\sharp$) where we use either the first or the second estimate depending on the smoothness of the initial condition and obtain a bounded sequence in $\mathscr{D}^{\nu,\beta}_{p,T}(\mathbb{R}^d, e^\sigma_l)$. The limit of this iteration (maybe after passing to a subsequence) is a local solution u, and as in [GP15b, Theorem 6.12]) those local solutions can be concatenated to a paracontrolled solution $u = F'(0)u \ll Y + u^\sharp \in \mathscr{D}^{\nu,\beta}_{p,T}(\mathbb{R}^d, e^\sigma_l)$ on $[0,T]$.

To verify uniqueness one can use that two different solutions $u = F'(0)u \ll Y + u^\sharp$, $v = F'(0)v \ll Y + v^\sharp$ for the same initial data have a difference $u - v = (u - v) \ll Y + (u^\sharp - v^\sharp)$ that solves once more the linear parabolic Anderson model with initial condition 0 so that the a priori estimates above give $u - v = 0$. $\square$

We can now deduce the main theorem of this section, where the parameters are as defined above.

Theorem 4.3.6 (Weak universality of PAM in dimension 2)**.** *Let u_0^ε be a uniformly bounded sequence in $\mathcal{C}^0_p(\mathcal{G}^\varepsilon, e^\sigma_l)$ such that $\mathscr{E}^\varepsilon u_0^\varepsilon$ converges to some u_0 in $\mathcal{S}'_\omega(\mathbb{R}^2)$. Then there are unique solutions $u^\varepsilon \in \mathscr{D}^{\alpha,\alpha}_{p,T^\varepsilon}(\mathcal{G}^\varepsilon, e^\sigma_l)$ to*

$$\mathscr{L}^\varepsilon_\mu u^\varepsilon = F^\varepsilon(u^\varepsilon)(\xi^\varepsilon - c^\varepsilon_\mu F'(0)), \qquad u^\varepsilon(0) = u_0^\varepsilon,$$

on $[0, T^\varepsilon)$ with $T^\varepsilon := T \wedge T^\varepsilon_{\mathrm{expl}}$ and $T^\varepsilon_{\mathrm{expl}} := \sup\{t \geqslant 0 | \|u^\varepsilon(t)\|_{\mathscr{D}^{\alpha,\alpha}_{p,T}} < \infty\}$. It holds $T^\varepsilon = T$ for ε small enough. The sequence $u^\varepsilon = F'(0)u^\varepsilon \ll Y^\varepsilon + u^{\varepsilon,\sharp} \in \mathscr{D}^{\alpha,\alpha}_{p,T}(\mathcal{G}^\varepsilon, e^\sigma_l)$ is uniformly bounded (for ε small enough such that $T = T^\varepsilon$). Their extensions $\mathscr{E}^\varepsilon u^\varepsilon$ converge in distribution in $\mathscr{D}^{\alpha,\alpha'}_{p,T}(\mathbb{R}^d, e^{\sigma'}_l)$, $\alpha' < \alpha$, $\sigma' < \sigma$, to the solution u of the linear equation in Corollary 4.3.5.

Remark 4.3.7. *Since T^ε is a random time the convergence in distribution has to be defined with some care: We say that $u^\varepsilon \to u$ in distribution if for any $f \in C_b(\mathscr{D}^{\alpha,\alpha'}_{p,T}(\mathcal{G}^\varepsilon, e^\sigma_l); \mathbb{R})$, which we extend to exploding paths by simply setting it to 0, we have $\mathbb{E}[f(u^\varepsilon)] = \mathbb{E}[f(u^\varepsilon)\mathbf{1}_{T^\varepsilon_{\mathrm{expl}} < T}] \to \mathbb{E}[f(u)]$ and further $\mathbb{P}(T^\varepsilon_{\mathrm{expl}} < T) \to 0$.*

Proof. Existence and uniform bounds for a solution u^ε follow similarly as in Corollary 4.3.5 with the only difference that, due to the presence of the quadratic term in the a priori estimates in Lemma 4.3.3, the time T^ε_* on which a Picard iteration can be set up is now of a more complicated form, namely

$$T^\varepsilon_* = C_1 M_\varepsilon^{-2 \cdot \frac{2}{\alpha - \delta}} \wedge C_2(u_0^\varepsilon) M_\varepsilon^{-4 \cdot \frac{2}{\alpha-\delta}} \varepsilon^{-\eta \cdot \frac{2}{\alpha - \delta}}, \tag{4.26}$$

where the first contribution, with a deterministic constant C_1 and M_ε from (4.20), comes from the linear part of the a priori estimate in Lemma 4.3.3 and the second contribution, with some deterministic polynomial C_2 in the initial condition

$\|u_0^\varepsilon\|_{\mathcal{C}_p^0(\mathcal{G}^\varepsilon, e_l^\sigma)}$, arises from the quadratic term. Using that ε is a dyadic we see by summing the $L^1(\mathbb{P})$ norm (or Borell-Cantelli) that the sequence $M_\varepsilon^4 \varepsilon^{\eta/2}$ is bounded almost surely, so that $M_\varepsilon^{-4\cdot\frac{2}{\alpha-\delta}} \varepsilon^{-\eta\cdot\frac{2}{\alpha-\delta}} = 1/(M_\varepsilon^4 \varepsilon^{\eta/2})^{\frac{2}{\alpha-\delta}} \cdot \varepsilon^{\frac{-\eta}{\alpha-\delta}}$ approaches ∞ almost surely as $\varepsilon \to 0$. Consequently for ε small enough we can set up the Picard iteration for a time of the form

$$T_*^\varepsilon = C_1 M_\varepsilon^{2\cdot\frac{2}{\alpha-\delta}}\,, \tag{4.27}$$

which is *independent* of the initial condition. Using (4.26) we can concatenate the paracontrolled solutions up to some time $T_{\mathrm{expl}}^\varepsilon \wedge T$, which due to (4.27) coincides with T for ε small enough.

To check the uniqueness of the discrete equation suppose that we are given two solutions $u^\varepsilon, v^\varepsilon$, which then satisfy

$$\begin{aligned}\mathscr{L}_\mu^\varepsilon(u^\varepsilon - v^\varepsilon) &= (F^\varepsilon(u^\varepsilon) - F^\varepsilon(v^\varepsilon))(\xi^\varepsilon - c_\mu^\varepsilon F'(0)) \\ &= \underbrace{\int_0^1 F'(u^\varepsilon + \zeta(v^\varepsilon - u^\varepsilon))\,\mathrm{d}\zeta}_{=:\mathscr{F}} \cdot (v^\varepsilon - u^\varepsilon)(\xi^\varepsilon - c_\mu^\varepsilon F'(0))\,.\end{aligned}$$

We already know, by the a priori estimates, that $u^\varepsilon = F'(0)u^\varepsilon \ll Y^\varepsilon + u^{\varepsilon,\sharp}$, $v^\varepsilon = F'(0)v^\varepsilon \ll Y^\varepsilon + v^{\varepsilon,\sharp}$ are bounded in $\mathscr{D}_{p,T_*^\varepsilon}^{\alpha,\alpha}(\mathcal{G}^\varepsilon, e_l^\sigma)$. As we only care now to prove uniqueness for a fixed scale ε we do not care about picking up negative powers of ε so that we can consider our equation started in "paracontrolled" initial conditions $u^\varepsilon(0) = v^\varepsilon(0) \in \mathcal{C}_p^\alpha(\mathcal{G}^\varepsilon, e_l^\sigma)$, $u^{\varepsilon,\sharp}(0) = v^{\varepsilon,\sharp}(0) \in \mathcal{C}_p^{2\alpha}(\mathcal{G}^\varepsilon, e_l^\sigma)$ and our solutions contained in $\mathscr{D}_{p,T_*^\varepsilon}^{0,\alpha}(\mathcal{G}^\varepsilon, e_l^\sigma)$. Consequently, since e_l^σ is an increasing function, the integral term $\mathscr{F}$ is an object in $L^\infty(\mathcal{G}^\varepsilon)$ and by picking up a further negative power of ε we can consider it as an element of $\mathcal{M}_{T_*^\varepsilon}^0 \mathcal{C}_\infty^\beta(\mathcal{G}^\varepsilon)$ for any $\beta \in \mathbb{R}$. The product $(v^\varepsilon - u^\varepsilon)(\xi^\varepsilon - c_\mu^\varepsilon F'(0))$ can be estimated as in the proof of Lemma 4.3.3. Since multiplication by $\mathscr{F}$ only contributes an (ε-dependent) factor we obtain a bound of the form

$$\|u^\varepsilon - v^\varepsilon\|_{\mathscr{D}_{p,T_*^\varepsilon}^{0,\alpha}} \lesssim_\varepsilon (T_*^\varepsilon)^{\frac{\alpha-\delta}{2}} \|u^\varepsilon - v^\varepsilon\|_{\mathscr{D}_{p,T_*^\varepsilon}^{0,\alpha}}\,,$$

which shows $\|u^\varepsilon - v^\varepsilon\|_{\mathscr{D}_{p,T}^{0,\alpha}} = 0$ for T_*^ε small enough. Iterating this argument gives $u^\varepsilon = v^\varepsilon$ on all of $[0, T^\varepsilon)$.

It remains to show that this unique solution $\mathscr{E}u^\varepsilon$ converges to u. By Skorohod representation we know that $\mathscr{E}^\varepsilon \xi^\varepsilon$, $\mathscr{E}^\varepsilon Y^\varepsilon$, $\mathscr{E}^\varepsilon(Y^\varepsilon \bullet \xi^\varepsilon)$ in Lemma 4.2.3 converge almost surely on a suitable probability space. We will work on this space from now on. The application of the Skorohod representation theorem is indeed allowed since the limiting measure of these objects has support in the closure of smooth functions and thus in a separable space. Having proved that the sequence u^ε is uniformly bounded in $\mathscr{D}^{\alpha,\alpha}_{p,T^\varepsilon}(\mathcal{G}^\varepsilon, e^\sigma_l)$ we know that $\mathscr{E}^\varepsilon u^\varepsilon$ is uniformly bounded in $\mathscr{D}^{\alpha;\alpha}_{p,T}(\mathbb{R}^d, e^\sigma_l)$ (for $\varepsilon > 0$ small enough such that $T^\varepsilon = T$). To show the convergence we note that by compact embedding arguments we obtain a convergent subsequence of $\mathscr{E}^\varepsilon u^\varepsilon$ that converges to some $u = F'(0)u \ll Y + u^\sharp \in \mathscr{D}^{\alpha,\alpha'}_{p,T}(\mathbb{R}^d, e^{\sigma'}_l)$ in distribution. If we can show that this limit solves

$$\mathscr{L}_\mu u = F'(0)u \blacklozenge \xi,\; u(0) = u_0 \tag{4.28}$$

for some white noise ξ, we can argue by uniqueness to finish the proof. We have

$$\mathscr{L}^\varepsilon_\mu \mathscr{E}^\varepsilon u^\varepsilon = \mathscr{E}^\varepsilon(F^\varepsilon(u^\varepsilon)(\xi^\varepsilon - c^\varepsilon_\mu F'(0)))\,,$$

where we already know, by considering the same decomposition as in Lemma 4.3.3, that the right hand side is bounded in $\mathcal{M}^\alpha_T \mathcal{C}^{2\alpha+2\kappa/\sigma-2}_p(\mathbb{R}^d, e^\sigma_l)$ and converges due to the ($\mathscr{E}$) property of the objects on the right hand side in distribution (in a weaker space) to $F'(0)u \blacklozenge \xi$. The convergence of the left hand side follows from Lemma 3.3.4. □

Since the weights we are working with are increasing, the solutions u^ε and the limit u are actually classical tempered distributions. However, since we need the $\mathcal{S}_\omega$ spaces to handle convolutions in e^σ_l-weighted spaces it is natural to allow for solutions in $\mathcal{S}'_\omega$. An exception is the case where ξ^ε is Gaussian, since then it can be handled by a logarithmic weight (compare Lemma 2.2.4) and therefore e^σ_l could be replaced by a time-dependent polynomial weight. In the linear case, $F = \mathrm{Id}$, we can allow for sub-exponentially growing initial conditions u_0 since the only reason for choosing the parameter l in the weight e^σ_{l+t} smaller than $-T$ was to be able to estimate $e^\sigma_{l+t} \leqslant (e^\sigma_{l+t})^2$ to handle the quadratic term. In this case the solution will be a genuine ultra-distribution.

Chapter 5

Interweaving Regularity structures and paracontrolled calculus

In Chapter 2 we found two distinct descriptions of the (anisotropic) Hölder-Zygmund spaces $\mathcal{C}_{\mathfrak{s}}^{\gamma}(\mathbb{R}^d) = \mathcal{B}_{\infty,\infty,\mathfrak{s}}^{\gamma}(\mathbb{R}^d)$ with scaling vector $\mathfrak{s} \in [1,\infty)^d$ and regularity $\gamma \in (0,\infty)\backslash|\mathbb{N}^d|_{\mathfrak{s}}$, given by Definition 2.1.16 and Lemma 2.1.23. Let us recall them by using the notion of the polynomial regularity structure $\mathscr{T} = (\overline{A}, \overline{\mathcal{T}}, \overline{G})$ with model $(\overline{\Pi}, \overline{\Gamma})$, introduced on page 50. Given $f \in \mathcal{C}_{\mathfrak{s}}^{\gamma}(\mathbb{R}^d)$ we can describe the statement of Lemma 2.1.23 in a concise way by introducing a modelled distribution $F : \mathbb{R}^d \to \overline{\mathcal{T}}$ given for $x \in \mathbb{R}^d$ by

$$F_x := \sum_{k \in \mathbb{N}^d_{<\gamma}} F_x^{X^k} X^k =: \sum_{\alpha \in \overline{A}^{<\gamma}} F_x^{\alpha} ,$$

where we wrote $F^{X^k} := \frac{1}{k!}\partial^k f$, $\overline{A}^{<\gamma} := \{|k|_{\mathfrak{s}} : | k \in \mathbb{N}^d_{<\gamma}\}$ and finally $F_x^{\alpha} := \sum_{k \in \mathbb{N}^d_{<\gamma} : |k|_{\mathfrak{s}} = \alpha} F_x^{X^k} X^k$ for the projection of F on the level $\alpha \in \overline{A}^{<\gamma}$. Lemma 2.1.23 can then be summarized as

$$\|F_y^{\alpha} - \overline{\Gamma}_{yx}^{\alpha} F_x\|_{\mathcal{T}_\alpha} \lesssim \|y - x\|_{\mathfrak{s}}^{\gamma - \alpha} \tag{5.1}$$

for $\alpha \in \overline{A}^{<\gamma}$, where we equipped the finite-dimensional space

$$\overline{\mathcal{T}}_{\alpha} = \operatorname{span}\left\{X^k : ||k|_{\mathfrak{s}} = \alpha\right\}$$

with some arbitrary norm $\|\cdot\|_{\mathcal{T}_\alpha}$. In Definition 2.1.16 however we introduced $\mathcal{C}_{\mathfrak{s}}^{\gamma}(\mathbb{R}^d)$ via Littlewood-Paley blocks Δ_j and the condition for $f \in \mathcal{C}_{\mathfrak{s}}^{\gamma}(\mathbb{R}^d)$ is formulated as $\|\Delta_j f\|_{L^\infty(\mathbb{R}^d)} \lesssim 2^{-j\gamma}$ or equivalently, by Lemma 2.1.33, for $\alpha \in \overline{A}^{<\gamma}$

$$\|\Delta_j F^{\alpha}\|_{L^\infty(\mathbb{R}^d; \mathcal{T}_\alpha)} \lesssim 2^{-j(\gamma - \alpha)} . \tag{5.2}$$

While (5.1) already implies $F^{X^k} = \frac{1}{k!}\partial^k F_0$ (or $F^\alpha = \sum_{|k|_\mathfrak{s}=\alpha} \frac{1}{k!}\partial^k F^0\, X^k$) and can thus serve as an alternative characterization of the space $\mathcal{C}_\mathfrak{s}^\gamma$, this is not true for (5.2). In fact, (5.2) does not pose any requirement at all on the interconnection between F^α and $F^{\alpha'}$ for $\alpha \neq \alpha'$. One therefore has to impose a condition such as $F^{X^{k+l}} = \partial^l F^{X^k}$ by hand, which can be also be written, using the operators $\overline{\Gamma}_{yx}$, as

$$(\partial^k F^\alpha)_x - (\partial^k \overline{\Gamma}^\alpha_{\cdot x} F_x)_x = \partial^k (F^\alpha - \overline{\Gamma}^\alpha_{\cdot x} F_x)_x = 0 \text{ for } k \in \mathbb{N}^d_{<\gamma-\alpha}\,, \tag{5.3}$$

where $(\partial^k \overline{\Gamma}^\alpha_{\cdot x} F_x)_x$ should be read as the derivative of the function $y \mapsto \Gamma^\alpha_{yx} F_x$ with respect to y, evaluated in the point x.

Criterion (5.1) is taken as the definition of the space of modelled distributions $\mathcal{D}^\gamma$ on a general regularity structure $\mathscr{T} = (A, \mathcal{T}, G)$ with model (Π, Γ), which we introduced in Definition 2.3.14. There is however no obvious way to generalize the Fourier description (5.2) to general $\mathscr{T}$. It was proposed already in [GIP15] to introduce a paraproduct $P(F, \Pi)$ on $\mathcal{T}$ and it was conjectured therein that it might be possible to describe the space $\mathcal{D}^\gamma$ via such objects. We here show that this is indeed the case by introducing a family of paraproducts $P(F, \Gamma^\alpha)$ and defining a space $\mathcal{P}^\gamma$ by requiring instead of (5.2)

$$\|\Delta_j (F^\alpha - P(F, \Gamma^\alpha))\|_{L^\infty} \lesssim 2^{-j(\gamma-\alpha)}\,, \tag{5.4}$$

(which is just saying $F^\alpha - P(F, \Gamma^\alpha) \in \mathcal{C}_\mathfrak{s}^{\gamma-\alpha}$) and the *structure condition* (5.3). Since the paraproducts $P(F, \Gamma^\alpha)$, described in Definition 5.1.1 below, vanish for F with components in the polynomial structure $\overline{\mathcal{T}}$, this is indeed a generalization of (5.2). In the main result of this chapter, Theorem 5.2.1 below, we then prove that indeed

$$\mathcal{D}^\gamma = \mathcal{P}^\gamma \tag{5.5}$$

so that (5.4) is a Fourier description of $\mathcal{D}^\gamma$, generalizing the Littlewood-Paley description of Hölder-Zygmund spaces.

In [HL17] the authors introduce a space $\mathcal{D}^\gamma_{p,q}(\mathbb{R}^d)$ which generalizes the characterization of Besov spaces $\mathcal{B}^\gamma_{p,q}(\mathbb{R}^d)$ via differences to the framework of modelled distributions (the special case $\mathcal{D}^\gamma_{p,p}(\mathbb{R}^d)$ was already introduced before by [PT16a]). The arguments below could probably be extended to this framework so that a Fourier description of these spaces could be given in terms of Littlewood-Paley blocks quite similar as in Definition 2.1.16 above. However, for the sake of simplicity we will only restrict ourselves to the case $p = q = \infty$ here.

Let us point out that one should always think of $\mathbb{R}^d$ in this chapter as incorporating space and time, so that we really introduce *space-time paraproducts* in contrast to the proceeding in Chapter 4, where paraproducts were only taken in the space variable.

In the context of SPDEs modelled distributions are actually defined on a finite-time interval only, i.e. on a set of the shape

$$(0,T)\times\mathbb{R}^{d-1}\,.$$

As the theory announced above only works with global objects, we need to introduce extension operators for modelled distributions, which we do in Proposition 5.3.13 and Theorem 5.3.16 below. In order to allow for possible blow-ups near the time $t=0$ we further introduce a space of singular modelled distributions $\mathcal{D}^{[\eta,\gamma]}$, which behaves similar as the space $\mathcal{D}^{\gamma,\eta}$ introduced in [Hai14], but is more suitable for our Fourier description.

For simplicity we only consider unweighted spaces in what follows, so that the ultra-distribution framework we introduced in Section 2.1 will not be used here.

This chapter will be content of [MP18].

A note on Banach valued distributions

In Definition 2.3.2 we defined a regularity structure to be a triple $\mathscr{T}=(A,\mathcal{T},G)$ with an index set A, a group G and a graded vector space

$$\mathcal{T}=\bigoplus_{\alpha\in A}\mathcal{T}_\alpha\,.$$

We allowed, following [Hai14], the space $\mathcal{T}_\alpha$ to be a Banach space, although "in practice" $\mathcal{T}_\alpha$ is typically finite-dimensional. The reason for this quite general choice is that in such a way the case of a Banach valued rough path setup (see for example [FV10]) can be implemented via a regularity structure, compare [Hai14, Section 4.4]. Although we do not have a specific application in mind, we don't want to restrict ourselves to forbid for such cases, especially since it is rather cheap to include them. The only price we have to pay is to consider vector valued distributions in this chapter.

Recall that for a Banach space X we can define the *Banach valued* Schwartz distributions $\mathcal{S}'(\mathbb{R}^d;X)$ to be the set of continuous linear functionals

$$f:\,\mathcal{S}(\mathbb{R}^d)\to X\,,$$

where $\mathcal{S}(\mathbb{R}^d)$ is just the classical space of (complex-valued) Schwartz functions. A similar construction can of course be made for tempered ultra-distributions as in Chapter 2.

A measurable functions $f:\mathbb{R}^d\to X$ such that

$$f(\varphi):=\int_{\mathbb{R}^d}\mathrm{d}x\,f(x)\varphi(x) \tag{5.6}$$

is well-defined for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ as a Bochner integral, can be identified via (5.6) with a distribution in $\mathcal{S}'(\mathbb{R}^d; X)$. Most concepts like support, differentiation, Fourier transform and multiplication with functions $\varphi \in \mathcal{S}(\mathbb{R}^d)$ carry over. The meaning of a concept should be apparent to a reader familiar to the concept of tempered distributions with values in $\mathbb{C}$, but if a doubt should arise one can consult for example [ST87] or[Tre75].

It is straightforward to repeat the construction in Chapter 2 to get *Banach valued* Besov spaces which we denote by $\mathcal{B}^{\gamma}_{p,q,\mathfrak{s}}(\mathbb{R}^d, \rho; X)$ by using the same *real valued* dyadic partition of unity. The Littlewood-Paley blocks $\Delta_j f \in C^\infty(\mathbb{R}^d; X)$ for $f \in \mathcal{S}'(\mathbb{R}^d; X)$ are then given by

$$\Delta_j f = \mathcal{F}^{-1}_{\mathbb{R}^d}\Big(\varphi_j \cdot \mathcal{F}_{\mathbb{R}^d} f\Big) = \int_{\mathbb{R}^d} \mathrm{d}u\, \Psi^j_{\cdot - u} \cdot f_u$$

where φ_j, $\Psi^j \in \mathcal{S}(\mathbb{R}^d; \mathbb{R})$ are as in Section 2.1 but now $\mathcal{F}_{\mathbb{R}^d} f$, $f \in \mathcal{S}'(\mathbb{R}^d; X)$ are Banach valued so that the integral on the right hand side should be read as a Bochner integral. Using $L^p(\mathbb{R}^d; X)$ to denote the Bochner spaces the norm for the unweighted space $\mathcal{B}^{\gamma}_{p,q}(\mathbb{R}^d; X)$ will then be given by

$$\|f\|_{\mathcal{B}^{\gamma}_{p,q}(\mathbb{R}^d;X)} := \left\| \left(2^{j\gamma} \|\Delta_j f\|_{L^p(\mathbb{R}^d;X)}\right)_{j \geqslant -1} \right\|_{\ell^q} .$$

However, in this chapter we will actually only work with the unweighted Hölder-Zygmund spaces $\mathcal{C}^{\gamma}_{\mathfrak{s}}(\mathbb{R}^d; X) = \mathcal{B}^{\gamma}_{\infty,\infty,\mathfrak{s}}(\mathbb{R}^d, 1; X)$. The only results from Chapter 2 which we cannot carry over immediately are the ones concerning interpolation (Lemma 2.1.31) and multiplication (Lemma 4.1.6 and Corollary 2.1.35). The multiplication rules are still valid if one of the factors takes values in the classical (complex valued) Besov spaces $\mathcal{B}^{\gamma}_{p,q,\mathfrak{s}}(\mathbb{R}^d, \rho) = \mathcal{B}^{\gamma}_{p,q,\mathfrak{s}}(\mathbb{R}^d, \rho; \mathbb{C})$ we defined in Chapter 2. This will always be the case in this chapter.

5.1 Paraproducts on a regularity structure

Let us start with a (simplified) repetition of the solution theory for the parabolic Anderson model on $[0, T] \times \mathbb{R}^2$, namely

$$(\partial_t - \Delta_{\mathbb{R}^2}) f = +f \cdot (\xi - \infty) \tag{5.7}$$

with periodic white noise $\xi \in \mathcal{S}'(\mathbb{R}^2)$. The idea in [GIP15], which we adapted in Chapter 4, is to define first $I\xi$ to be the time-independent solution to $(\partial_t - \Delta_{\mathbb{R}^2}) I\xi = -\Delta_{\mathbb{R}^2} I\xi = \xi$ (denoted by Y in Chapter 4) and to consider instead of f the object $f^{\#} := f - f \prec I\xi$ with the paraproduct

$$(f \prec I\xi)_x = \sum_{j \geqslant -1} \sum_{i < j-1} (\Delta_i f)_x (\Delta_j I\xi)_x = \sum_{j>0} \iint \mathrm{d}u\mathrm{d}v\, \Psi^{<j-1}_{x-u} \Psi^j_{x-v}\, f_u \cdot (I\xi)_v\,, \tag{5.8}$$

where the integration domain for each integral should be read as *space-time*, that is $\mathbb{R}^{1+2}$. Hence, we cheat a little bit since in Chapter 4 we rather take a modified paraproduct $\ll$ in space. It then turns out that $f^\#$ solves a "better" equation than f, which allows to derive a priori estimates and to solve the equation. In (5.8) we now take functions Ψ^j, $\Psi^{<j-1}$ that are constructed as in Section 2.1 with an *anisotropic* scaling $\mathfrak{s}$, more precisely we take in (5.8) the parabolic scaling $\mathfrak{s} = \mathfrak{s}_{\mathrm{par}} = (2,1,1)$, which is one more disparity compared to Chapter 4 or [GIP15].

In [Hai14] the problem (5.7) is solved on a regularity structure (again with $\mathfrak{s} = \mathfrak{s}_{\mathrm{par}}$) with a model space spanned by the symbol set $\{\mathcal{I}(\Xi)\} \cup \{X^k : k \in \mathbb{N}^d\}$ and equipped with a model (Π, Γ) such that

$$\Pi_x X^k = \Gamma^{\mathbf{1}}_{yx} X^k = (y-x)^k,\ \Pi_x \mathcal{I}(\Xi)(y) = \Gamma^{\mathbf{1}}_{yx} \mathcal{I}\Xi = I\xi(y) - I\xi(x)\,.$$

The solution f to (5.7) is represented by a modelled distribution F that turns out to take the form

$$F = f\,\mathbf{1} + f\,\mathcal{I}(\Xi) + \sum_{|k|_{\mathfrak{s}_{\mathrm{par}}=1}} f^{X^k} X^k\,, \tag{5.9}$$

where f^{X^k} are some real valued functions. Recalling from Lemma 2.1.14 that Ψ^j integrates polynomials (and constants) to 0 for $j > 0$ we can express (5.8) via F and the model (Π, Γ):

$$(f \prec I\xi)_x = \sum_{j>0} \int \mathrm{d}u\mathrm{d}v\, \Psi^{<j-1}_{x-u} \Pi_u F_u (\Psi^j_{x-\cdot}) = \sum_{j>0} \iint \mathrm{d}u\mathrm{d}v\, \Psi^{<j-1}_{x-u} \Psi^j_{x-v} \Gamma^{\mathbf{1}}_{vu} F_u \tag{5.10}$$

This motivates the following definitions.

Definition 5.1.1. *Let $\mathscr{T} = (A, \mathcal{T}, G)$ be a regularity structure with some scaling vector $\mathfrak{s}$ and let Ψ^j, $\Psi^{<j-1} \in \mathcal{S}(\mathbb{R}^d)$ be functions as in (2.17) constructed from some anisotropic dyadic partition of unity with the same $\mathfrak{s}$. Given a model (Π, Γ) on $\mathscr{T}$ we define the following* paraproducts

$$P(F, \Pi)_x = \sum_{j>0} \int \mathrm{d}u \Psi^{<j-1}_{x-u}\, \Pi_u F_u (\Psi^j_{x-\cdot}) \tag{5.11}$$

$$P(F, \Gamma^\alpha)_x = \sum_{j>0} \iint \mathrm{d}u\mathrm{d}v\, \Psi^{<j-1}_{x-u} \Psi^j_{x-v}\, \Gamma^\alpha_{vu} F_u \tag{5.12}$$

for any $F : \mathbb{R}^d \to \mathcal{T}$ and $\alpha \in A$ for which this is defined. Both identities should be read in $\mathcal{S}'(\mathbb{R}^d)$ and are written in formal notation.

Remark 5.1.2. *If $\mathcal{T}_\alpha$ is finite dimensional for some $\alpha \in A$ and we have a basis $\{e_i\}$ for $\mathcal{T}_\alpha$ we will also write*

$$P(F, \Gamma^{e_i}) := \sum_{j>0} \iint \mathrm{d}u\mathrm{d}v \, \Psi^{<j-1}_{x-u} \Psi^j_{x-v} \, \Gamma^{e_i}_{vu} F_u \, ,$$

where we recall that $\Gamma^{e_i}_{vu} F_u$ denotes the coefficient of $\Gamma^\alpha_{vu} F_u \in \mathcal{T}_\alpha$ in front of e_i. In particular we have

$$P(F, \Gamma^\alpha) = \sum_i P(F, \Gamma^{e_i}) \, e_i \, .$$

For instance, assume that $\mathcal{T}$ satisfies Assumption 2.3.12. We then have for $\alpha \in |\mathbb{N}^d|_{\mathfrak{s}} = \overline{A} \subseteq A$

$$P(F, \Gamma^\alpha) = \sum_{k \in \mathbb{N}^d \colon |k|_{\mathfrak{s}} = \alpha} P(F, \Gamma^{X^k}) \, X^k \, ,$$

where we took $\{X^k \,|\, |k|_{\mathfrak{s}} = \alpha\}$ as a basis for $\mathcal{T}_\alpha = \overline{\mathcal{T}}_\alpha$. For $\alpha = 0$ we simply have

$$P(F, \Gamma^0) = P(F, \Gamma^{\mathbf{1}}) \cdot \mathbf{1} \, .$$

For measurable, at most polynomially growing F the expressions (5.11) and (5.12) are well-defined. Indeed: Each of the terms in the sums in (5.11) and (5.12) is spectrally supported in a $2^{j\mathfrak{s}}$ scaled rectangular annulus as in Definition 2.1.11. This can be easily checked for smooth $(x, y) \mapsto \Gamma_{yx} F_x$, $\Pi_x F_x(y)$, so that the general case follows by approximation. Further by Definition 2.3.9 and Lemma 2.1.14 one easily sees that each of the terms can be bounded by $2^{-j\gamma}$ for some $\gamma \in \mathbb{R}$, with $\gamma > 0$ in case of $P(F, \Gamma^\alpha)$ and $\gamma = \min A$ for $P(F, \Pi)$. Lemma 2.1.19 then shows that both paraproducts are containted in some Besov space $\mathcal{C}^\gamma_{\mathfrak{s}}(\mathbb{R}^d; X) \subseteq \mathcal{S}'(\mathbb{R}^d; X)$. Since γ is positive for (5.12) the object $P(F, \Gamma^\alpha)$ is in fact a (slightly) Hölder continuous function, while $P(F, \Pi)$ can be a genuine distribution.

Although we have in (5.10) the identity $P(F, \Gamma^{\mathbf{1}}) = P(F, \Pi)$ this is not true in general and these two paraproducts will play quite different roles in the theory presented here. But it is useful to keep in mind that $P(F, \Gamma^{\mathbf{1}}) = P(F, \Pi)$ as soon as F is *function like*, that is F takes values in a sector of non-negative regularity.

Let's come back to our toy example above. The function f^{X^k} from (5.9) did not appear in the approach in [GIP15], which apparently lies in the fact that polynomials are erased in the paraproduct (5.10). Since our goal is to find a link between the ideas in [GIP15] and [Hai14] we need some extra ingredient that forces the f^{X^k} to enter the game, this will be the task of the *structure condition*, which we already motivated in (5.3).

Definition 5.1.3. *Let $\mathscr{T} = (A, \mathcal{T}, G)$ be a regularity structure with a model* (Π, Γ), *$\mathcal{V} \subseteq \mathcal{T}$ be a sector and let $\mathcal{V}\backslash\mathcal{W} \subseteq \mathcal{V}$ be the complement of a sector within $\mathcal{V}$ as in Definition 2.3.7. Let further $\Omega \subseteq \mathbb{R}^d$ be open. We say that $F : \mathbb{R}^d \to \mathcal{T}$ satisfies the structure condition on Ω for $\gamma \in \mathbb{R}$ and $\alpha \in A_{\mathcal{V}\backslash\mathcal{W}}$ with $\alpha < \gamma$ if for any $x \in \Omega$ and $k \in \mathbb{N}^d$ with $|k|_{\mathfrak{s}} \leqslant \gamma - \alpha$ the map $v \mapsto \partial^k \Psi^{\leqslant N}_{x-v}(F^\alpha_v - \Gamma^\alpha_{vx} F_x)$ is in $L^1(\mathbb{R}^d)$ for $N \in \mathbb{N}$ large enough and if*

$$\lim_{N\to\infty} \int \mathrm{d}v\, \partial^k \Psi^{\leqslant N}_{x-v}(F^\alpha_v - \Gamma^\alpha_{vx} F_x) = 0\,, \tag{5.13}$$

where the limit is taken in $\mathcal{T}_\alpha$. We say that the structure condition holds on Ω for γ and $A_{\mathcal{V}\backslash\mathcal{W}}$ if the condition is satisfied on Ω for γ and all $\alpha \in A_{\mathcal{V}\backslash\mathcal{W}}$ with $\alpha < \gamma$

Remark 5.1.4. *Condition* (5.13) *translates for smooth F, Γ into*

$$\partial^k(F^\alpha - \Gamma^\alpha_{\cdot x} F_x)_x = 0\,, \tag{5.14}$$

which is just the identity we announced in (5.3)*.*

We are now ready to give a notion of a paracontrolled distribution that is comparable with the space $\mathcal{D}^\gamma$ of modelled distributions.

Definition 5.1.5. *Let $\gamma \in \mathbb{R}$ and let $\mathscr{T} = (A, \mathcal{T}, G)$ be a regularity structure with a model (Π, Γ). Let $\mathcal{V}\backslash\mathcal{W}$ be the complement of a sector $\mathcal{W}$ within a sector $\mathcal{V} \subseteq \mathcal{T}$ as in Definition 2.3.7.*

We say that $F : \mathbb{R}^d \to \mathcal{V}^-_\gamma$ with $F \in C_b(\mathbb{R}^d; \mathcal{V}\backslash\mathcal{W})$ is in $\mathcal{P}^\gamma(\mathbb{R}^d; \mathcal{V}\backslash\mathcal{W}) = \mathcal{P}^\gamma(\mathbb{R}^d; \mathcal{V}\backslash\mathcal{W}, \Gamma)$ if for $\alpha \in A_{\mathcal{V}\backslash\mathcal{W}}$

$$F^{\sharp,\alpha} := F^\alpha - P(F, \Gamma^\alpha) \in \mathcal{C}^{\gamma-\alpha}_{\mathfrak{s}}(\mathbb{R}^d; \mathcal{T}_\alpha) \tag{5.15}$$

and if the structure condition (5.13) *is fullfilled on $\mathbb{R}^d$ for γ and $A_{\mathcal{V}\backslash\mathcal{W}}$. If $\mathcal{W} = \{0\}$ we write $\mathcal{P}^\gamma(\mathbb{R}^d; \mathcal{V}) := \mathcal{P}^\gamma(\mathbb{R}^d; \mathcal{V}\backslash\{0\})$.*

We define the semi-norm

$$\|f\|_{\mathcal{P}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\mathcal{W})} := \|F\|_{C_b(\mathbb{R}^d;\mathcal{V}\backslash\mathcal{W})} + \sup_{\alpha\in A_{\mathcal{V}\backslash\mathcal{W}}} \|F^{\sharp,\alpha}\|_{\mathcal{C}^{\gamma-\alpha}_{\mathfrak{s}}(\mathbb{R}^d;\mathcal{T}_\alpha)}\,.$$

and define for models (Π, Γ), $(\hat{\Pi}, \hat{\Gamma})$ and $F \in \mathcal{P}^\gamma(\mathbb{R}^d; \mathcal{V}\backslash\mathcal{W}, \Gamma)$, $\hat{F} \in \mathcal{P}^\gamma(\mathbb{R}^d; \mathcal{V}\backslash\mathcal{W}, \hat{\Gamma})$ the "distance"

$$\|F; \hat{F}\|_{\mathcal{P}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\mathcal{W},\Gamma,\hat{\Gamma})} := \|F - \hat{F}\|_{C_b(\mathbb{R}^d;\mathcal{V}\backslash\mathcal{W})} + \sup_{\alpha\in A_{\mathcal{V}\backslash\mathcal{W}}} \|F^{\sharp,\alpha} - \hat{F}^{\sharp,\alpha}\|_{\mathcal{C}^{\gamma-\alpha}_{\mathfrak{s}}(\mathbb{R}^d;\mathcal{T}_\alpha)}$$

If $F \in \mathcal{P}^\gamma(\mathbb{R}^d; \mathcal{V})$ we say that a distribution $f \in \mathcal{S}'(\mathbb{R}^d) = \mathcal{S}'(\mathbb{R}^d; \mathbb{C})$ is paracontrolled *by $F \in \mathcal{P}^\gamma(\mathbb{R}^d; \mathcal{V})$ if*

$$f^\sharp := f - P(F, \Pi) \in \mathcal{C}^\gamma_{\mathfrak{s}}(\mathbb{R}^d)\,. \tag{5.16}$$

Remark 5.1.6. ⚠ *We have the same remark as for Definition 2.3.14: The notation "$\mathcal{V}\backslash\mathcal{W}$" in $F \in \mathcal{P}^\gamma(\mathbb{R}^d; \mathcal{V}\backslash\mathcal{W})$ indicates two things, first that F takes values in the sector $\mathcal{V}$ (and* ***not only*** *in $\mathcal{V}\backslash\mathcal{W}$) and moreover that* (5.15) *(and the structure condition) is true for its components in $\mathcal{V}\backslash\mathcal{W}$. In particular we have*

$$\mathcal{P}^\gamma(\mathbb{R}^d; \mathcal{V}) \subseteq \mathcal{P}^\gamma(\mathbb{R}^d; \mathcal{V}\backslash\mathcal{W})\,.$$

Remark 5.1.7. *Since for $\alpha \in A_{\mathcal{V}\backslash\mathcal{W}}$ and $x, y \in \mathbb{R}^d$ we have $\Gamma^\alpha_{yx} F_x = \sum_{\alpha' \in A_{\mathcal{V}\backslash\mathcal{W}} : \alpha' \geqslant \alpha} \Gamma^\alpha_{yx} F^{\alpha'}_x$ the paraproduct $P(F, \Gamma^\alpha)$ is really independent of the components of F which are not in $\mathcal{V}\backslash\mathcal{W}$. As we did require F to have bounded components in $\mathcal{V}\backslash\mathcal{W}$ we see that the paraproducts $P(F, \Gamma^\alpha)$ in* (5.15) *are well-defined.*

In [GIP15] an operator like $P(F, \Pi)$ as well as a notion like (5.16) were already introduced. The authors further give an alternative construction for the reconstruction operator $\mathcal{R}$ (compare Theorem 2.3.19) based on Littlewood-Paley theory, with $\mathcal{R}$ satisfying a slightly weaker bound. By uniqueness these two objects coincide however for $\gamma > 0$, see [GIP15, Lemma 6.7]. The authors also show the following result, which we will use occasionally.

Lemma 5.1.8. *Given $F \in \mathcal{D}^\gamma(\mathbb{R}^d; \mathcal{T})$ with $\gamma > 0$ the distribution $\mathcal{R}F$ is paracontrolled by F, more precisely*

$$\|\mathcal{R}F - P(F, \Pi)\|_{\mathcal{C}^\gamma_\mathfrak{s}(\mathbb{R}^d;\mathbb{R})} \lesssim \|F\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{T})}\,.$$

As we already pointed out above, what we presented in Definition 5.1.1 is not a strict generalization of the approach from [GIP15] or Chapter 4 as we use instead of a modified paraproduct $\lll$, as in Definition 4.1.4, a space-time paraproduct (5.8). A construction in space-time might seem more natural, but we have to pay a price: Since the solutions to SPDEs such as (5.7) are only defined on a compact time interval, we have to extend them in order to make sense of the space-time paraproduct. We postpone the presentation of the corresponding tools and their application to Sections 5.3 and 6.2 and turn now to the main result of this chapter. With the help of the operators $P(F, \Gamma^\alpha)$ and the structure condition (5.13) we are now able to give a complete correspondence between the concepts from paracontrolled analysis (as modified above) and regularity structures, this will be the content of Theorem 5.2.1 in the next section.

5.2 Modelled distributions are paracontrolled

The following theorem is the main result of this chapter, it reveals a deep connection between the paraproduct approach and the theory of regularity structures. We show that the spaces $\mathcal{D}^\gamma(\mathbb{R}^d, \mathcal{V}\backslash\mathcal{W})$ and $\mathcal{P}^\gamma(\mathbb{R}^d; \mathcal{V}\backslash\mathcal{W})$ are identical. For technical reasons we have to exclude the case that $\gamma \in \mathbb{R}$ is contained in the locally finite set

$$A_{\mathbb{N}^d} := A + |\mathbb{N}^d|_{\mathfrak{s}} \tag{5.17}$$

This is necessary since we want to apply for the spaces $\mathcal{C}_{\mathfrak{s}}^{\gamma-\alpha}$ appearing in the Definition of 5.1.5 the Hölder characterization from Lemma 2.1.23. If one sees the following theorem as a generalization of Lemma 2.1.23 then the exclusion of (5.17) corresponds to the restriction $\gamma \notin |\mathbb{N}^d|_{\mathfrak{s}}$ required there.

Theorem 5.2.1. *Let $\mathscr{T} = (A, \mathcal{T}, G)$ be a regularity structure with a subsector $\mathcal{V}$ and a complement $\mathcal{V}\backslash\mathcal{W}$ of a subsector $\mathcal{W} \subseteq \mathcal{V}$ within $\mathcal{V}$. Let further (Π, Γ) be a model on $\mathscr{T}$. We then have for any $\gamma \in \mathbb{R}\backslash A_{\mathbb{N}^d}$*

$$\mathcal{D}^\gamma(\mathbb{R}^d; \mathcal{V}\backslash\mathcal{W}, \Gamma) = \mathcal{P}^\gamma(\mathbb{R}^d; \mathcal{V}\backslash\mathcal{W}, \Gamma) \tag{5.18}$$

with equivalent norms (where the equivalence constants can be chosen proportional to some polynomial in $\|\Gamma\|_\gamma$). Moreover, given a second model $(\hat{\Pi}, \hat{\Gamma})$ and modelled distributions $F \in \mathcal{D}^\gamma(\mathbb{R}^d; \mathcal{V}\backslash\mathcal{W}, \Gamma)$, $\hat{F} \in \mathcal{D}^\gamma(\mathbb{R}^d; \mathcal{V}\backslash\mathcal{W}, \hat{\Gamma})$ we also have

$$\begin{aligned}
\|F; \hat{F}\|_{\mathcal{P}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\mathcal{W},\Gamma,\hat{\Gamma})} &\lesssim K_1 \cdot \|F; \hat{F}\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\mathcal{W},\Gamma,\hat{\Gamma})} \\
\|F; \hat{F}\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\mathcal{W},\Gamma,\hat{\Gamma})} &\lesssim K_1 \cdot \|F; \hat{F}\|_{\mathcal{P}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\mathcal{W},\Gamma,\hat{\Gamma})} \\
&\quad + \|\Gamma - \hat{\Gamma}\|_\gamma (\|F\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\mathcal{W},\Gamma} + \|\hat{F}\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\mathcal{W},\hat{\Gamma})}),
\end{aligned}$$

where $K_1 > 0$ is a polynomial in the norms of F, $\hat{F}$, Γ and $\hat{\Gamma}$.

We have further the following local embedding property *for polynomials: Suppose the regularity structure satisfies Assumption* (2.3.12) *and we are given an open set Ω and a map $F + P : \mathbb{R}^d \to \mathcal{V}_-^\gamma$ that satisfies the structure condition* (5.13) *on Ω for γ and $\alpha \in A_\mathcal{V} \cap |\mathbb{N}^d|_{\mathfrak{s}}$ with $\alpha < \gamma$. If F, P are chosen such that $F : \mathbb{R}^d \to (\mathcal{V}\backslash\overline{\mathcal{T}})_-^\gamma$, $P : \mathbb{R}^d \to \overline{\mathcal{T}}_-^\gamma$, we have*

$$\|F + P\|_{\mathcal{D}^\gamma(\Omega;\mathcal{V})} \lesssim K_2 \cdot \Big[\|F\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\overline{\mathcal{T}})} \tag{5.19}$$

$$+ \sup_{\alpha \in A_\mathcal{V} \cap |\mathbb{N}^d|_{\mathfrak{s}}} \Big(\|P^\alpha\|_{C_b(\Omega;\mathcal{T}_\alpha)} + \|P^\alpha - P(F, \Gamma^\alpha)\|_{\mathcal{C}_{\mathfrak{s}}^{\gamma-\alpha}(\Omega;\mathcal{T}_\alpha)} \Big) \Big], \tag{5.20}$$

where $K_2 > 0$ is some polynomial in $\|F\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\overline{\mathcal{T}})}$, $\|P^\alpha\|_{C_b(\Omega;\mathcal{T}_\alpha)}$, $\|\Gamma\|_\gamma$ Moreover given two such functions $F + P$, $\hat{F} + \hat{P}$ for the models (Π, Γ) and $(\hat{\Pi}, \hat{\Gamma})$ we have

$$\begin{aligned}
\|(F + P); (\hat{F} + \hat{P})\|_{\mathcal{D}^\gamma(\Omega;\mathcal{V},\Gamma,\hat{\Gamma})} &\lesssim K_3 \cdot \Big[\|F; \hat{F}\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\overline{\mathcal{T}},\Gamma,\hat{\Gamma})} + \|\Gamma - \hat{\Gamma}\|_\gamma \\
+ \sup_{\alpha \in A_\mathcal{V} \cap |\mathbb{N}^d|_{\mathfrak{s}}} \Big(\|P^\alpha - \hat{P}^\alpha\|_{C_b(\Omega;\mathcal{T}_\alpha)} &+ \|P^\alpha - \hat{P}^\alpha - P(F - \hat{F}, \Gamma^\alpha)\|_{\mathcal{C}_{\mathfrak{s}}^{\gamma-\alpha}(\Omega;\mathcal{T}_\alpha)} \Big) \Big],
\end{aligned}$$

where $K_3 > 0$ *is some polynomial in* $\|F\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\overline{\mathcal{T}},\Gamma)}$, $\|\hat{F}\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\overline{\mathcal{T}},\hat{\Gamma})}$, $\|P^\alpha\|_{C_b(\Omega;\mathcal{T}_\alpha)}$, $\|P^\alpha - P(F,\Gamma^\alpha)\|_{\mathcal{C}_\mathfrak{s}^{\gamma-\alpha}(\Omega;\mathcal{T}_\alpha)}$, $\|\hat{P}^\alpha\|_{C_b(\Omega;\mathcal{T}_\alpha)}$, $\|\hat{P}^\alpha - P(F,\hat{\Gamma}^\alpha)\|_{\mathcal{C}_\mathfrak{s}^{\gamma-\alpha}(\Omega;\mathcal{T}_\alpha)}$, $\|\Gamma\|_\gamma$ *and* $\|\hat{\Gamma}\|_\gamma$.

Proof. We will include polynomial powers of $\|\Gamma\|_\gamma$ in the constant indicated by "$\lesssim$". We assume without loss of generality that $A_{\mathcal{V}\backslash\mathcal{W}}$ contains only elements below γ.

To show $\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\mathcal{W}) \subseteq \mathcal{P}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\mathcal{W})$ note first that $F \in \mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\mathcal{W})$ already implies the structure condition (5.13) on $\mathbb{R}^d$ for γ and $\alpha \in A_{\mathcal{V}\backslash\mathcal{W}}$ since for $k \in \mathbb{N}^d$ with $|k|_\mathfrak{s} < \gamma - \alpha$ one has

$$\left\| \int \mathrm{d}v\ \partial^k \Psi^{<N}_{x-v}(F^\alpha_v - \Gamma^\alpha_{vx} F_x) \right\|_{\mathcal{T}_\alpha} \overset{\text{Lemma 2.1.14}}{\lesssim} \|F\|_{\mathcal{D}^\gamma(\overline{W})} \cdot 2^{N|k|_\mathfrak{s}} \cdot 2^{-N(\gamma-\alpha)} \overset{N\to\infty}{\longrightarrow} 0\,.$$

We then follow similar ideas as in [GIP15, Subsection 6.2.]: We can rewrite for $x \in \mathbb{R}^d$ and $\alpha \in A_{\mathcal{V}\backslash\mathcal{W}}$

$$F^\alpha_x - P(F,\Gamma^\alpha)_x = \sum_{j>0} \int \mathrm{d}v \left(\Psi^j_{x-v} F^\alpha_v - \int \mathrm{d}u \Psi^{<j-1}_{x-u}\, \Psi^j_{x-v} \Gamma^\alpha_{vu} F_u \right) + (\Delta_{\leqslant 0} F^\alpha)_x\,.$$

As $\Delta_{\leqslant 0} F^\alpha = \Psi^{\leqslant 0} * F^\alpha$ is smooth with bounded derivatives we only have to consider the first term on the right hand side. Since the spectral support of the summands is contained in an annulus scaled by $2^{j\mathfrak{s}}$ it is sufficient to bound each term by $2^{-j(\gamma-\alpha)}\|F\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\mathcal{W})}$ due to Lemma 2.1.19. And indeed we have, using $\int \Psi^{<j-1}_{x-u} 1 = 1$ (Lemma 2.1.14),

$$\int \mathrm{d}v \left(\Psi^j_{x-v} F^\alpha_v - \int \mathrm{d}u \Psi^{<j-1}_{x-u} \Psi^j_{x-v} \Gamma^\alpha_{vu} F_u \right) = \iint \mathrm{d}u\mathrm{d}v\ \Psi^{<j-1}_{x-u} \Psi^j_{x-v} (F^\alpha_v - \Gamma^\alpha_{vu} F_u)\,.$$

Now, by assumption

$$\|F^\alpha_v - \Gamma^\alpha_{vu} F_u\|_{\mathcal{T}_\alpha} \lesssim \|F\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\mathcal{W})} \|u-v\|^{\gamma-\alpha}_\mathfrak{s} \lesssim \|F\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\mathcal{W})} (\|u-x\|^{\gamma-\alpha}_\mathfrak{s} + \|v-x\|^{\gamma-\alpha}_\mathfrak{s})\,,$$

so that we have with Lemma 2.1.14

$$\| \iint \mathrm{d}u\mathrm{d}v\ \Psi^{<j-1}_{x-u} \Psi^j_{x-v} (F^\alpha_v - \Gamma^\alpha_{vu} F_u) \|_{\mathcal{T}_\alpha} \lesssim \|F\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\mathcal{W})} 2^{-j(\gamma-\alpha)}\,,$$

which proves $\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\mathcal{W}) \subseteq \mathcal{P}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\mathcal{W})$.

Let's now adress the delicate direction of the proof, that is $\mathcal{P}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\mathcal{W}) \subseteq \mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\mathcal{W})$. We assume without loss of generality that $\|F\|_{\mathcal{P}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\mathcal{W})} \leqslant 1$ and show by induction in $A_{\mathcal{V}\backslash\mathcal{W}}$ that for $x, y \in \mathbb{R}^d$ and $\alpha \in A_{\mathcal{V}\backslash\mathcal{W}}$

$$\|F^\alpha_y - \Gamma^\alpha_{yx} F_x\|_{\mathcal{T}_\alpha} \lesssim \|y-x\|^{\gamma-\alpha}_\mathfrak{s}\,.$$

Note that it is sufficient to take $\|x-y\|_{\mathfrak{s}} \leqslant 1$, compare Remark 2.3.16. Let us start our induction with $\alpha = \max A_{\mathcal{V}\setminus\mathcal{W}}$ (which exists since we assumed $\max A_{\mathcal{V}\setminus\mathcal{W}} < \gamma < \infty$). By definition of Γ_{yx} we have $\Gamma^{\alpha}_{yx} F_x = F^{\alpha}_x$ and thus $P(F,\Gamma^{\alpha}) = 0$ due to Lemma 2.1.14. Hence $F^{\alpha} = F^{\sharp,\alpha} \in \mathcal{C}^{\gamma-\alpha}_{\mathfrak{s}}(\mathbb{R}^d)$ and via 5.13 we obtain that for $k \in \mathbb{N}^d$, $0 < |k|_{\mathfrak{s}} < \gamma - \alpha$ (if any) $\partial^k F = 0$. Thus

$$
\begin{aligned}
\|F^{\alpha}_y - \Gamma^{\alpha}_{yx} F_x\|_{\mathcal{T}_\alpha} &= \|F^{\alpha}_y - F^{\alpha}_x\|_{\mathcal{T}_\alpha} = \|F^{\alpha}_y - F^{\alpha}_x - \sum_{k \in \mathbb{N}^d_{<\gamma-\alpha}} \partial^k F^{\alpha}_x \,(y-x)^k\|_{\mathcal{T}_\alpha} \\
&= \|F^{\sharp,\alpha}_y - F^{\sharp,\alpha}_x - \sum_{k \in \mathbb{N}^d_{<\gamma-\alpha}} \partial^k F^{\sharp,\alpha}_x \,(y-x)^k\|_{\mathcal{T}_\alpha} \lesssim \|y-x\|^{\gamma-\alpha}_{\mathfrak{s}} ,
\end{aligned}
$$

where we applied Lemma 2.1.20 (together with $\|y-x\|_{\mathfrak{s}} \leqslant 1$).

Let us now assume that we already know for some $\alpha \in A_{\mathcal{V}\setminus\mathcal{W}}$ that for any $\alpha' \in A_{\mathcal{V}\setminus\mathcal{W}}$, $\alpha' > \alpha$

$$
\|F^{\alpha'}_y - \Gamma^{\alpha'}_{yx} F_x\|_{\mathcal{T}_{\alpha'}} \lesssim \|y-x\|^{\gamma-\alpha}_{\mathfrak{s}} . \tag{5.21}
$$

We then show that (5.21) does also hold for all $\alpha' = \alpha$. To this end we reshape

$$
\begin{aligned}
&F^{\alpha}_y - \Gamma^{\alpha}_{yx} F_x = F^{\alpha}_y - F^{\alpha}_x - \sum_{\alpha' \in A_{\mathcal{V}\setminus\mathcal{W}} \colon \alpha' > \alpha} \Gamma^{\alpha}_{yx} F^{\alpha'}_x \\
&= F^{\sharp,\alpha}_y - F^{\sharp,\alpha}_x - \sum_{k \colon 0 < |k|_{\mathfrak{s}} < \gamma - \alpha} \frac{1}{k!} \partial^k F^{\sharp,\alpha}_x (y-x)^k && (5.22) \\
&+ P(F,\Gamma^{\alpha})_y - P(F,\Gamma^{\alpha})_x + \sum_{k \colon 0 < |k|_{\mathfrak{s}} < \gamma - \alpha} \frac{1}{k!} \partial^k \left(F^{\alpha} - P(F,\Gamma^{\alpha})\right)_x (y-x)^k && (5.23) \\
&- \sum_{\alpha' \in A_{\mathcal{V}\setminus\mathcal{W}} \colon \alpha' > \alpha} \Gamma^{\alpha}_{yx} F^{\alpha'}_x .
\end{aligned}
$$

Since (5.22) does already decay in the right order due to Lemma 2.1.23 (and the assumption $\gamma \notin A_{\mathbb{N}^d}$), we are only left with the last line which we identify as the limit for $N \to \infty$ (in $\mathcal{T}_\alpha$ for every x, y) of

$$
\begin{aligned}
D^N := \sum_{j \leqslant N} D^N_j := \sum_{j \leqslant N} \Bigg[\int \mathrm{d}w \left(\Psi^j_{y-w} - \sum_{k \in \mathbb{N}^d_{<\gamma-\alpha}} \frac{1}{k!} \partial^k \Psi^j_{x-w} (y-x)^k \right) \\
\times \left(P(F,\Gamma^{\alpha})_w - \sum_{\alpha' \in A_{\mathcal{V}\setminus\mathcal{W}} \colon \alpha' > \alpha} \Gamma^{\alpha}_{wx} F^{\alpha'}_x \right) \Bigg] .
\end{aligned}
$$

Indeed: observe the following three limits

$$\sum_{j\leqslant N}\int \mathrm{d}w\left(\Psi^j_{y-w}-\Psi^j_{x-w}\right)P(F,\Gamma^\alpha)_w \tag{5.24}$$
$$\overset{N\to\infty}{\longrightarrow} P(F,\Gamma^\alpha)_y-P(F,\Gamma^\alpha)_x\,,$$

$$-\sum_{j\leqslant N,\,0<|k|_{\mathfrak{s}}<\gamma-\alpha}\ \sum_{0<|k|_{\mathfrak{s}}<\gamma-\alpha}\int \mathrm{d}w\,\partial^k\Psi^j_{x-w}\Big(P(F,\Gamma^\alpha)_w-\sum_{\alpha'\in A_{\mathcal{V}\setminus\mathcal{W}}\colon\alpha'>\alpha}\Gamma^\alpha_{wx}F^{\alpha'}_x\Big)(y-x)^k \tag{5.25}$$
$$\overset{N\to\infty}{\longrightarrow}\sum_{k\colon 0<|k|_{\mathfrak{s}}<\gamma-\alpha}\frac{1}{k!}\partial^k\left(F^\alpha-P(F,\Gamma^\alpha)\right)_x(y-x)^k\,,$$

$$\sum_{j\leqslant N}\int \mathrm{d}w\left(\Psi^j_{y-w}-\Psi^j_{x-w}\right)\sum_{\alpha'\in A_{\mathcal{V}\setminus\mathcal{W}}\colon\alpha'>\alpha}\Gamma^\alpha_{wx}F^{\alpha'}_x \tag{5.26}$$
$$\overset{N\to\infty}{\longrightarrow}\sum_{\alpha'\in A_{\mathcal{V}\setminus\mathcal{W}}\colon\alpha'>\alpha}\Gamma^\alpha_{yx}F^{\alpha'}_x-\sum_{\alpha'\in A_{\mathcal{V}\setminus\mathcal{W}}\colon\alpha'>\alpha}\Gamma^\alpha_{xx}F^{\alpha'}_x=\sum_{\alpha'\in A_{\mathcal{V}\setminus\mathcal{W}}\colon\alpha'>\alpha}\Gamma^\alpha_{yx}F^{\alpha'}_x\,,$$

where we used (5.13) for (5.25) and the continuity of $\Gamma^\alpha_{\cdot x}F^{\alpha'}_x$ in (5.26) [1]. Writing $D_N=(5.24)+(5.25)+(5.26)$ we see the claimed convergence of D_N. Note, that we can reshape

$$\begin{aligned}
P(F,\Gamma^\alpha)_w &= \sum_{j>0}\iint \mathrm{d}u\mathrm{d}v\,\Psi^{<j-1}_{w-u}\Psi^j_{w-v}\Gamma^\alpha_{vu}F_u\\
&= \sum_{j>0}\iint \mathrm{d}u\mathrm{d}v\,\Psi^{<j-1}_{w-u}\Psi^j_{w-v}\Gamma^\alpha_{vu}(F_u-\Gamma_{ux}F_x)+(\varDelta_{>0}\Gamma^\alpha_{\cdot x}F_x)_w\\
&= \sum_{j>0}\sum_{\alpha'\in A_{\mathcal{V}\setminus\mathcal{W}}\colon\alpha'>\alpha}\iint \mathrm{d}u\mathrm{d}v\,\Psi^{<j-1}_{w-u}\Psi^j_{w-v}\Gamma^\alpha_{vu}(F^{\alpha'}_u-\Gamma^{\alpha'}_{ux}F_x)\\
&\quad+\sum_{\alpha'\in A_{\mathcal{V}\setminus\mathcal{W}}\colon\alpha'>\alpha}(\varDelta_{>0}\Gamma^\alpha_{\cdot x}F^{\alpha'}_x)_w\,,
\end{aligned}$$

where we used in the last line for both terms that $\int\Psi^j_v 1=0$ for $j>0$ to cancel the $\alpha'=\alpha$ components. We can therefore reshape D^N_j as (with $R^{\gamma-\alpha}_{x;y-x}$ being the Taylor

[1] A short computation shows that Definition 2.3.9 already implies (Hölder) continuity of the maps $y\mapsto\Gamma^\alpha_{yx}\tau$ for $\tau\in\mathcal{T}$ and $x\in\mathbb{R}^d$.

remainder as in (2.22))

$$D_j^N = \sum_{j\leqslant N}\sum_{\alpha'\in A_{\mathcal{V}\setminus\mathcal{W}}:\,\alpha'>\alpha}\int \mathrm{d}w\, R^{\gamma-\alpha}_{x-w;y-x}\Psi^j \tag{5.27}$$

$$\times\left[\sum_{i>0}\iint \mathrm{d}u\mathrm{d}v\,\Psi^{<i-1}_{w-v}\Psi^i_{w-u}\Gamma^\alpha_{vu}(F^{\alpha'}_u-\Gamma^{\alpha'}_{ux}F_x)+(\mathit{\Delta}_{\leqslant 0}\Gamma^\alpha_{\cdot x}F^{\alpha'}_x)_w\right]$$

$$=\sum_{j\leqslant N}\sum_{\alpha'\in A_{\mathcal{V}\setminus\mathcal{W}}:\,\alpha'>\alpha}\int \mathrm{d}wR^{\gamma-\alpha}_{x-w;y-x}\Psi^j\left[\sum_{i>0:\,i\sim j}\iint \mathrm{d}u\mathrm{d}v\,\Psi^{<i-1}_{w-v}\Psi^i_{w-u}\Gamma^\alpha_{vu}(F^{\alpha'}_u-\Gamma^{\alpha'}_{ux}F_x)\right] \tag{5.28}$$

$$+\sum_{\alpha'\in A_{\mathcal{V}\setminus\mathcal{W}}:\,\alpha'>\alpha}\int \mathrm{d}w\,\Psi^{<N+1}_w\cdot R^{\gamma-\alpha}_{x-w;y-x}(\Delta_{\leqslant 0}\Gamma^\alpha_{\cdot x}F^{\alpha'}_x)\,,$$

where we used in the second line spectral support properties to restrict the inner sum to $i\sim j$ and the convolution-like structure to move in the last term the Taylor remainder to the Littlewood-Paley block. The last term can be estimated by $\|x-y\|^{\gamma-\alpha}_{\mathfrak{s}}$ via Lemma 2.1.20 and 2.1.14 if one uses that for $k\in\mathbb{N}^d$ there is a $C>0$ such that

$$\|\partial^k\left(\mathit{\Delta}_{\leqslant 0}\Gamma^\alpha_{\cdot x}F^{\alpha'}_x\right)(w-x+v^k_t(y-x))\|_{\mathcal{T}_\alpha}\lesssim(1+\|w\|^C_{\mathfrak{s}})\,, \tag{5.29}$$

which can be easily checked by direct computation. To handle the term (5.28) we first estimate the sum in the square brackets by the induction hypothesis and Lemma 2.1.14:

$$\sum_{i:\,i\sim j}\sum_{\alpha'\in A_{\mathcal{V}\setminus\mathcal{W}}:\,\alpha'>\alpha}\left\|\iint \mathrm{d}u\mathrm{d}v\,\Psi^{<i-1}_{w-v}\Psi^i_{w-u}\Gamma^\alpha_{vu}(F^{\alpha'}_u-\Gamma^{\alpha'}_{ux}F_x)\right\|_{\mathcal{T}_\alpha} \tag{5.30}$$

$$\lesssim\sum_{\alpha'\in A_{\mathcal{V}\setminus\mathcal{W}}:\,\alpha'>\alpha}2^{-j(\alpha'-\alpha)}2^{-j(\gamma-\alpha')}\lesssim 2^{-j(\gamma-\alpha)}\,. \tag{5.31}$$

The rest of the estimate for (5.28) then follows via the exact same proceeding as in Lemma 2.1.23: Pick j_0 such that $2^{-j_0-1}<\|x-y\|_{\mathfrak{s}}\leqslant 2^{-j_0}$ and bound the sum up to a constant factor by

$$\sum_{j\leqslant j_0}\sum_{k\in\mathbb{N}^d_{>\gamma-\alpha}}\|x-y\|^{|k|_{\mathfrak{s}}}_{\mathfrak{s}}\,2^{j(|k|_{\mathfrak{s}}-(\gamma-\alpha))}+\sum_{N\geqslant j>j_0}\sum_{k\in\mathbb{N}^d_{<\gamma-\alpha}}\|x-y\|^{|k|_{\mathfrak{s}}}_{\mathfrak{s}}2^{-j(\gamma-\alpha-|k|_{\mathfrak{s}})}$$

$$\lesssim\|x-y\|^{\gamma-\alpha}_{\mathfrak{s}}\,,$$

where we applied Lemma 2.1.20 in the low-frequency case and in both cases the scaling of Ψ^j from Lemma 2.1.14. In total

$$\|F^\alpha_y-\Gamma^\alpha_{yx}F_x\|_{\mathcal{T}_\alpha}\lesssim\|x-y\|^{\gamma-\alpha}_{\mathfrak{s}}\,,$$

which closes the induction and finishes the proof of (5.18). For (5.19) we are in the case $\mathcal{W} = \{0\}$, so that $\mathcal{V}\backslash\mathcal{W} = \mathcal{V}$. Concerning (5.19) it remains to show that one has for $\alpha \in A_{\mathcal{V}} \cap |\mathbb{N}^d|_{\mathfrak{s}}$ and $x, y \in \Omega$ the bound

$$\|F_y^\alpha - \Gamma_{yx}^\alpha F_x\|_{\mathcal{T}_\alpha} \lesssim \|y - x\|^{\gamma-\alpha}\Big(\|F\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\overline{\mathcal{T}})} + \sup_{\alpha \in A_{\mathcal{V}} \cap |\mathbb{N}^d|_{\mathfrak{s}}} \sup_{\alpha \in A_{\mathcal{V}} \cap |\mathbb{N}^d|_{\mathfrak{s}}} \|P^\alpha - P(F, \Gamma^\alpha)\|_{\mathcal{C}_{\mathfrak{s}}^{\gamma-\alpha}(\Omega)}\Big) .$$

We can repeat almost the same proof as above: Note that the only cases where we used $F \in \mathcal{D}^\gamma$ were (5.22), (5.29) and (5.31). While it is now enough to recall in (5.22) and (5.29) that now $x \in \Omega$, we can restrict the sum in (5.31) to $\alpha' \in A_{\mathcal{V}}\backslash|\mathbb{N}^d|_{\mathfrak{s}}$ since polynomial entries vanish in the paraproduct (Lemma 2.1.14) and (5.19) is proved.

The distance estimates for models (Π, Γ), $(\hat{\Pi}, \hat{\Gamma})$ follow almost immediately from a repetition of the computations performed above, one only has to repeat the induction (5.21) for $\|F_y^{\alpha'} - \Gamma_{yx}^{\alpha'} F_x - (\hat{F}_y^{\alpha'} - \hat{\Gamma}_{yx}^{\alpha'} \hat{F}_x)\|$ instead. □

5.3 Singular spaces and extensions

5.3.1 Singular modelled distributions

Throughout this subsection we will fix a regularity structure $\mathscr{T} = (A, \mathcal{T}, G)$ (not necessarily satisfying Assumption 2.3.12) together with a scaling vector

$$\mathfrak{s} = (\theta, \mathfrak{s}') \in (1, \infty) \times [1, \infty)^{d-1}$$

for some $\theta > 1$. We further consider a time horizon $T \in (0, 1]$ and an associated family of sets

$$\Omega_t^T := (t, T) \times \mathbb{R}^{d-1},\ \Omega^T := (0, T) \times \mathbb{R}^{d-1} = \bigcup_{t \in (0,T)} \Omega_t^T$$

for $t \in (0, T)$. We will also, similarly as in [Hai14], introduce a set

$$P^T := \left\{x \in \mathbb{R}^d \ :\ x_1 \in \{0, T\}\right\} .$$

so that $\Omega^T = \big([0, T] \times \mathbb{R}^d\big) \backslash P^T$. The reason why we also exclude points $x_1 = T$ is only technical and lies in the fact that we prefer to work on open sets.

In [Hai14, Definition 6.2] the author defines the notion of a singular, modelled distribution $F : \mathbb{R}^d \to \mathcal{V}_-^\gamma \in \mathcal{D}^{\gamma,\eta}(\Omega^T; \mathcal{T})$, where "typically" $\eta \leqslant \gamma$, with norm

$$\|F\|_{\mathcal{D}^{\gamma,\eta}(\Omega^T;\mathcal{T})} := \sup_{t \in (0,1)} \sup_{\alpha \in A} t^{\frac{(\alpha-\eta)\vee 0}{\theta}} \sup_{x \in \Omega_t^T} \|F_x^\alpha\|_{\mathcal{T}_\alpha} + \sup_{t \in (0,1)} \sup_{\alpha \in A} \sup_{x,y \in \Omega_t^T, \|y-x\|_{\mathfrak{s}} \leqslant t} t^{\frac{\gamma-\eta}{\theta}} \frac{\|F_y^\alpha - \Gamma_{yx}^\alpha F_x\|_{\mathcal{T}_\alpha}}{\|y - x\|^{\gamma-\alpha}} . \tag{5.32}$$

In fact, the norm introduced in [Hai14] has a slightly different definition, but is equivalent to the expression in (5.32). $\mathcal{D}^{\gamma,\eta}(\Omega^T;\mathcal{T})$ should be seen as the space of modelled distribution with a possible blow-up at $\{x_1 = 0\}$. This space then shows nice behavior under multiplication and composition with smooth functions [Hai14, Proposition 6.12, 6.13]. It further has the following property for $\beta \in [\eta,\gamma]$, $\alpha < \beta$ and $x, y \in \Omega_t^T$, $\|y-x\|_{\mathfrak{s}} \leqslant t$

$$\|F_y^\alpha - \Gamma_{yx}^\alpha F_x^{<\beta}\|_{\mathcal{T}_\alpha} \lesssim \|F\|_{\mathcal{D}^{\gamma,\eta}(\Omega^T;\mathcal{T})}\, t^{\frac{\eta-\beta}{\theta}} \|y-x\|^{\beta-\alpha}\,, \tag{5.33}$$

where $F^{<\beta} := \sum_{\alpha' \in A:\, \alpha<\beta} F^{\alpha'}$, which means that "lower regularity comes with a weaker weight". Alas, the space $\mathcal{D}^{\gamma,\eta}(\Omega^T;\mathcal{T})$ is useless for our purposes due to the locality condition

$$\|y-x\| \leqslant t \tag{5.34}$$

for $y, x \in \Omega^T$ in the second term in (5.32). Since a Fourier based approach is highly non-local a requirement like (5.34) is hard to translate. We here propose to require instead 5.33 from the beginning (for general points not only those satisfying (5.34)). The estimates for multiplication and composition are then again true, compare Lemma 5.3.7 and 5.3.9 below (except for a slight increase in the weight which is irrelevant for our purposes).

Definition 5.3.1. *Let $\mathcal{V} \subseteq \mathcal{T}$ be a sector, let $\mathcal{V}\backslash\mathcal{W}$ be the complement of a sector $\mathcal{W}$ within $\mathcal{V}$ and let (Π,Γ) be a model on $\mathcal{T}$. Given parameters η, $\gamma \in \mathbb{R}\backslash A_{\mathbb{N}^d}$ with $\eta \leqslant \gamma$ we say that $F : \Omega^T \to \mathcal{V}$ is an element of $\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\mathcal{W}) = \mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\mathcal{W},\Gamma)$ if the following semi-norm is finite*

$$\|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\mathcal{W})} := \sup_{\beta \in [\eta,\gamma]\backslash A_{\mathbb{N}^d}} \sup_{t \in (0,T)} t^{\frac{\beta-\eta}{\theta}} \|F^{<\beta}\|_{\mathcal{D}^\beta(\Omega_t^T;\mathcal{V}\backslash\mathcal{W})} < \infty\,, \tag{5.35}$$

where $F^{<\beta} := \sum_{\alpha \in A:\, \alpha<\beta} F^\alpha$. Given two models (Π,Γ), $(\hat\Pi,\hat\Gamma)$ and $F \in \mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\mathcal{W},\Gamma)$, $\hat F \in \mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\mathcal{W},\hat\Gamma)$ we also introduce the notion of a "distance"

$$\|F;\hat F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\mathcal{W},\Gamma,\hat\Gamma)} := \sup_{\beta \in [\eta,\gamma]\backslash A_{\mathbb{N}^d}} \sup_{t \in (0,T)} t^{\frac{\beta-\eta}{\theta}} \|F^{<\beta};\hat F^{<\beta}\|_{\mathcal{D}^\beta(\Omega_t^T;\mathcal{V}\backslash\mathcal{W},\Gamma,\hat\Gamma)} < \infty\,. \tag{5.36}$$

If $\mathcal{W} = \{0\}$ we simply write $\mathcal{D}^{[\eta,\gamma]}(\Omega;\mathcal{V}) := \mathcal{D}^{[\eta,\gamma]}(\Omega;\mathcal{V}\backslash\{0\})$, a similar remark applies for the distance (5.36).

Remark 5.3.2. ⚠ *A remark such as 2.3.15 applies: The notation "$\mathcal{V}\backslash\mathcal{W}$" in $F \in \mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\mathcal{W})$ is supposed to indicate two things, first that F takes values in the sector $\mathcal{V}$ (and* ***not only*** *in $\mathcal{V}\backslash\mathcal{W}$) and moreover that* (5.35) *is finite for its components in $\mathcal{V}\backslash\mathcal{W}$. In particular we have*

$$\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}) \subseteq \mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\mathcal{W})\,.$$

Note that a slight technical difference to (2.3.14) *is that we allow a priori for F with values above γ, so that* (5.35) *is really only a semi-norm, even if $\mathcal{W} = \{0\}$.*

Remark 5.3.3. *The exclusion of the set $A_{\mathbb{N}^d}$ is actually not needed at this stage of our theory, it will however be convenient in Section 6.2 below.*

Note that we do not require any lower bound for η so that β in the supremum in (5.35) might be below the lowest regularity of $\mathcal{V}$ and in this case we simply have $F^{<\beta} = 0$. We further extend by convention $F \in \mathcal{D}^{[\eta,\gamma]}(\Omega^T)$ to $\mathbb{R}^d$ by setting $F(x) = 0$ for $x \in (\Omega^T)^c$. If $F \in \mathcal{D}^{[\eta,\gamma]}(\Omega^T)$ we have in fact also a bound in $\mathcal{D}^\beta$ for $\beta < \eta$ as shown by the subsequent lemma.

Lemma 5.3.4. *Let $F \in \mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\mathcal{W})$ with $\mathcal{V}\backslash\mathcal{W}, \eta, \gamma$ as in Definition 5.3.1. We then have for $t \in (0,T)$ and $\beta \leqslant \gamma$*

$$\|F^{<\beta}\|_{\mathcal{D}^\beta(\Omega_t^T;\mathcal{V}\backslash\mathcal{W})} \lesssim (1 + \|\Gamma\|_\eta)\, t^{\frac{(\eta-\beta)\wedge 0}{\theta}}\, \|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\mathcal{W})}$$

with $F^{<\beta}$ as in Definition 5.3.1.

Proof. For $\beta \geqslant \eta$ this is just the Definition of $\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\mathcal{W})$ (without even the need of the constant $(1 + \|\Gamma\|_\eta)$). For the case $\beta < \eta$ note first that $F^{<\eta} \in \mathcal{D}^\eta(\Omega^T;\mathcal{V}\backslash\mathcal{W})$ with $\|F\|_{\mathcal{D}^\eta(\Omega^T;\mathcal{V}\backslash\mathcal{W})} \leqslant \|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\mathcal{W})}$. Remark 3.2 of [Hai14] then states that $F^{<\beta} = (F^{<\eta})^{<\beta} \in \mathcal{D}^\beta(\Omega^T;\mathcal{V}\backslash\mathcal{W})$ with

$$\|F^{<\beta}\|_{\mathcal{D}^\beta(\Omega^T;\mathcal{V}\backslash\mathcal{W})} \lesssim (1 + \|\Gamma\|_\eta)\|F^{<\eta}\|_{\mathcal{D}^\eta(\Omega^T;\mathcal{V}\backslash\mathcal{W})}$$

and the claim follows. □

The following lemma shows that the first term in (5.32) is bounded for $F \in \mathcal{D}^{[\eta,\gamma]}$ (up to an arbitrarily small loss $\kappa > 0$).

Lemma 5.3.5. *Let $F \in \mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\mathcal{W})$ with $\mathcal{V}\backslash\mathcal{W}, \eta, \gamma$ as in Definition 5.3.1. We then have for $\alpha \in A_{\mathcal{V}\backslash\mathcal{W}}$, $t \in (0,T)$, $x \in \Omega_t^T$ and any $\kappa > 0$*

$$\|F_x^\alpha\|_{\mathcal{T}_\alpha} \lesssim t^{\frac{(\eta-\alpha-\kappa)\wedge 0}{\theta}}\|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\mathcal{W})}\,. \tag{5.37}$$

Given a second model $(\hat\Pi,\hat\Gamma)$ and $\hat F \in \mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\mathcal{W})$ we also have

$$\|F_x^\alpha - \hat F_x^\alpha\|_{\mathcal{T}_\alpha} \lesssim t^{\frac{(\eta-\alpha-\kappa)\wedge 0}{\theta}}\|F;\hat F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\mathcal{W},\Gamma,\hat\Gamma)}\,. \tag{5.38}$$

Proof. For both inequalities, (5.37) and (5.38), take $\beta = (\alpha + \kappa) \vee \eta$ in (5.35) or (5.36) respectively, with $\kappa > 0$ without loss of generality small enough such that $\beta \in [\eta, \gamma] \backslash A_{\mathbb{N}^d}$. The claim then follows due to $\eta - (\alpha + \kappa) \vee \eta = (\eta - \alpha - \kappa) \wedge (\eta - \eta) = (\eta - \alpha - \kappa) \wedge 0$. □

Lemma 5.3.5 implies in particular that $\mathcal{D}^{[\eta,\gamma]}(\Omega^T; \mathcal{T}) \subseteq \mathcal{D}^{\gamma,\eta-\kappa}(\Omega^T; \mathcal{T})$, so that we can apply results like [Hai14, Proposition 6.9] to have a reconstruction $\mathcal{R}F$ for $F \in \mathcal{D}^{[\eta,\gamma]}(\Omega^T; \mathcal{T})$, when F is extended by 0 as explained above.

Before we show that our space $\mathcal{D}^{[\eta,\gamma]}(\Omega^T; \mathcal{T})$ essentially behaves like $\mathcal{D}^{\gamma,\eta}(\Omega^T; \mathcal{T})$ under multiplication let us recall the definition of a product on $\mathscr{T}$.

Definition 5.3.6 (Definition 4.1, 4.6 from [Hai14])**.** *A product $\star$ is a continuous bilinear map from $\mathcal{T} \times \mathcal{T}$ to $\mathcal{T}$ such that $\tau_1 \star \tau_2 \in \mathcal{T}_{\alpha_1+\alpha_2}$ for $\tau_1 \in \mathcal{T}_{\alpha_1}$, $\tau_2 \in \mathcal{T}_{\alpha_2}$ and $\alpha_1, \alpha_2 \in A$, where we set $\mathcal{T}_{\alpha_1+\alpha_2} = \{0\}$ if $\alpha_1 + \alpha_2 \notin A$.*

A pair of sectors $\mathcal{V}_1$, $\mathcal{V}_2 \subseteq \mathcal{T}$ is called γ-regular for $\gamma \in \mathbb{R}$ if for any $\Gamma \in G$, $\tau_1 \in \mathcal{V}_{\alpha_1}$, $\tau_2 \in \mathcal{V}_{\alpha_2}$ with $\alpha_1, \alpha_2 \in A$, $\alpha_1 + \alpha_2 < \gamma$ we have $\Gamma(\tau_1 \star \tau_2) = \Gamma\tau_1 \star \Gamma\tau_2$.

Lemma 5.3.7. *Let $\mathcal{V}_1$, $\mathcal{V}_2 \subseteq \mathcal{T}$ be two sectors of regularity α_1, α_2 and let $\eta_1, \gamma_1, \eta_2, \gamma_2 \in \mathbb{R} \backslash A_{\mathbb{N}^d}$ with $\eta_1 \leqslant \gamma_1$ and $\eta_2 \leqslant \gamma_2$ be such that $\gamma = (\gamma_1 + \alpha_2) \wedge (\gamma_2 + \alpha_1) \notin A_{\mathbb{N}^d}$ and let $\kappa > 0$ be such that $\eta = \gamma \wedge (\eta_1 + \eta_2 - \kappa) \wedge (\eta_1 + \alpha_2 - \kappa) \wedge (\eta_2 + \alpha_1 - \kappa) \notin A_{\mathbb{N}^d}$. If the pair $(\mathcal{V}_1, \mathcal{V}_2)$ is γ-regular, and we are given some model (Π, Γ) and maps $F \in \mathcal{D}^{[\eta_1,\gamma_1]}(\Omega^T; \mathcal{V}_1, \Gamma)$, $G \in \mathcal{D}^{[\eta_2,\gamma_2]}(\Omega^T; \mathcal{V}_2, \Gamma)$, then we have the estimate*

$$\|F \star G\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{T})} \lesssim (1 + \|\Gamma\|_{\gamma_1 \vee \gamma_2}) \cdot \|F\|_{\mathcal{D}^{[\eta_1,\gamma_1]}(\Omega^T;\mathcal{V}_1)} \|G\|_{\mathcal{D}^{[\eta_2,\gamma_2]}(\Omega^T;\mathcal{V}_2)} \,. \tag{5.39}$$

If we are given a second model $(\hat{\Pi}, \hat{\Gamma})$ and $\hat{F} \in \mathcal{D}_{\alpha_1}^{[\eta_1,\gamma_1]}(\Omega^T; \mathcal{V}_1, \hat{\Gamma})$, $\hat{G} \in \mathcal{D}_{\alpha_2}^{[\eta_2,\gamma_2]}(\Omega^T; \mathcal{V}_2, \hat{\Gamma})$, we also have

$$\begin{aligned} \|F \star G; \hat{F} \star \hat{G}\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{T})} \lesssim K \cdot \big(& \|F; \hat{F}\|_{\mathcal{D}^{[\eta_1,\gamma_1]}(\Omega^T;\mathcal{V}_1,\Gamma,\hat{\Gamma})} \\ & + \|G; \hat{G}\|_{\mathcal{D}^{[\eta_2,\gamma_2]}(\Omega^T;\mathcal{V}_2,\Gamma,\hat{\Gamma})} + \|\Gamma - \hat{\Gamma}\|_{\gamma_1 \vee \gamma_2} \big) \,, \end{aligned} \tag{5.40}$$

where K is a polynomial in the corresponding norms of F, $\hat{F}$, G, $\hat{G}$, Γ and $\hat{\Gamma}$.

Proof. Note that, without loss of generality, we can choose $\kappa > 0$ smaller than any $\varepsilon > 0$ and we can assume that $\|F\|_{\mathcal{D}^{[\eta_1,\gamma_1]}(\Omega^T;\mathcal{V}_1)} \leqslant 1$ and $\|G\|_{\mathcal{D}^{[\eta_2,\gamma_2]}(\Omega^T;\mathcal{V}_2)} \leqslant 1$. Fix $\beta \in [\eta, \gamma] \backslash A_{\mathbb{N}^d}$ and $\alpha \in A$ with $\alpha < \beta$.

Consider for $t \in (0, T)$ and $x, y \in \Omega_t^T$

$$\begin{aligned} \Gamma_{yx}^{\alpha} (F \star G)_x^{<\beta} - (F \star G)_y^{\alpha} &= \Gamma_{yx}^{\alpha} \left(\sum_{\nu_1+\nu_2<\beta} F_x^{\nu_1} \star G_x^{\nu_2} \right) - \sum_{\mu_1+\mu_2=\alpha} F_y^{\mu_1} \star G_y^{\mu_2} \\ &= \sum_{\mu_1+\mu_2=\alpha} \left(\sum_{\nu_1+\nu_2<\beta} \Gamma_{yx}^{\mu_1} F_x^{\nu_1} \star \Gamma_{yx}^{\mu_2} G_x^{\nu_2} - F_y^{\mu_1} \star G_y^{\mu_2} \right) , \end{aligned}$$

where the sums run over $\nu_1 \in A_{\mathcal{V}_1}$, $\nu_2 \in A_{\mathcal{V}_2}$ and $\mu_1 \in A_{\mathcal{V}_1}$, $\mu_2 \in A_{\mathcal{V}_2}$. We will bound each term on the right hand side separately. Let us reshape first

$$\sum_{\nu_1+\nu_2<\beta} \Gamma^{\mu_1}_{yx} F^{\nu_1}_x \star \Gamma^{\mu_2}_{yx} G^{\nu_2}_x = \sum_{\nu_1<\beta-\mu_2} \Gamma^{\mu_1}_{yx} F^{\nu_1}_x \star \Gamma^{\mu_2}_{yx} G^{<\beta-\nu_1}_x ,$$

where we used that the terms on the right hand side vanish as soon as $\beta - \nu_1 \leqslant \mu_2 \Leftrightarrow \nu_1 \geqslant \beta - \mu_2$. Our first task is to estimate

$$\sum_{\nu_1<\beta-\mu_2} \Gamma^{\mu_1}_{yx} F^{\nu_1}_x \star \left(\Gamma^{\mu_2}_{yx} G^{<\beta-\nu_1}_x - G^{\mu_2}_y\right) \tag{5.41}$$

and then

$$\left(\sum_{\nu_1<\beta-\mu_2} \Gamma^{\mu_1}_{yx} F^{\nu_1}_x - F^{\mu_1}_y\right) \star G^{\mu_2}_y = \left(\Gamma^{\mu_1}_{yx} F^{<\beta-\mu_2}_x - F^{\mu_1}_y\right) \star G^{\mu_2}_y . \tag{5.42}$$

We first bound (5.41) in $\|\cdot\|_{\mathcal{T}_\alpha}$ by applying Lemma 5.3.5 and (2.38) for the first and Lemma 5.3.4 for the second factor. Up to a constant proportional to $(1+\|\Gamma\|_{\gamma_1})$ this yields

$$\begin{aligned} &\|y-x\|_{\mathfrak{s}}^{\nu_1-\mu_1} t^{\frac{(\eta_1-\nu_1-\kappa)\wedge 0}{\theta}} \cdot \|y-x\|_{\mathfrak{s}}^{\beta-\nu_1-\mu_2} t^{\frac{(\eta_2-(\beta-\nu_1))\wedge 0}{\theta}} \\ &= \|y-x\|_{\mathfrak{s}}^{\beta-\alpha} t^{\frac{(\eta_1-\nu_1-\kappa)\wedge 0+(\eta_2-(\beta-\nu_1))\wedge 0}{\theta}} \end{aligned} \tag{5.43}$$

Note that the application of Lemma 5.3.4 was allowed since $\nu_1 \geqslant \alpha_1$ and thus

$$\beta - \nu_1 \leqslant \gamma - \nu_1 \leqslant \gamma - \alpha_1 = (\gamma_1+\alpha_2) \wedge (\gamma_2+\alpha_1) - \alpha_1 \leqslant (\gamma_2+\alpha_1) - \alpha_1 = \gamma_2$$

Let us bound the exponent of t in (5.43) from below, by distinguishing between the 4 different cases that might occur

$$\begin{aligned} &(\eta_1-\nu_1-\kappa) \wedge 0 + (\eta_2-(\beta-\nu_1)) \wedge 0 \\ &= \begin{cases} 0 \\ \eta_1-\nu_1-\kappa \\ \eta_2+\nu_1-\beta \\ \eta_1+\eta_2-\kappa-\beta \end{cases} \geqslant \begin{cases} 0 \\ \eta_1+\mu_2-\kappa-\beta \\ \eta_2+\nu_1-\kappa-\beta \\ \eta_1+\eta_2-\kappa-\beta \end{cases} \geqslant \eta-\beta , \end{aligned}$$

where we used $\nu_1 < \beta - \mu_2$ in the first and $\mu_2 \geqslant \alpha_2$ as well as $\nu_1 \geqslant \alpha_1$ in the second inequality. Using $t \in (0,T) \subseteq (0,1)$ we can thus bound (5.43) as

$$\|y-x\|_{\mathfrak{s}}^{\beta-\alpha} t^{\frac{(\eta_1-\nu_1-\kappa)\wedge 0+(\eta_2-(\beta-\nu_1))\wedge 0}{\theta}} \leqslant \|y-x\|_{\mathfrak{s}}^{\beta-\alpha} t^{\frac{\eta-\beta}{\theta}} .$$

Applying once more Lemma 5.3.4 and 5.3.5 the term (5.42) can be bounded, up to a constant proportional to $(1 + \|\Gamma\|_{\gamma_2})$, as

$$\|y-x\|^{\beta-\mu_2-\mu_1}\, t^{\frac{(\eta_1-(\beta-\mu_2))\wedge 0}{\theta}}\, t^{\frac{(\eta_2-\mu_2-\kappa)\wedge 0}{\theta}} = \|y-x\|^{\gamma-\alpha}\, t^{\frac{(\eta_1-(\beta-\mu_2))\wedge 0+(\eta_2-\mu_2-\kappa)\wedge 0}{\theta}}\,.$$

The exponent of t can be bounded from below as above, so that we get altogether

$$\|(F\star G)^\alpha_y - \Gamma^\alpha_{yx}(F\star G)^{<\beta}_x\|_{\mathcal{T}_\alpha} \lesssim (1+\|\Gamma\|_{\gamma_1\vee\gamma_2})\, \|y-x\|_{\mathfrak{s}}^{\beta-\alpha}\, t^{\frac{\eta-\beta}{\theta}}\,.$$

To estimate the first term of the norm (2.46) note that for α and β as above we have by Lemma 5.3.5 for $x\in\Omega^T_t$

$$\|(F\star G)^\alpha_x\|_{\mathcal{T}_\alpha} = \Big\|\sum_{\mu_1+\mu_2=\alpha} F^{\mu_1}_x \star G^{\mu_2}_x\Big\|_{\mathcal{T}_\alpha} \lesssim \sum_{\mu_1+\mu_2=\alpha} t^{\frac{(\eta_1-\mu_1-\kappa/2)}{\theta}\wedge 0}\cdot t^{\frac{\eta_2-\mu_2-\kappa/2}{\theta}\wedge 0} \lesssim t^{\eta-\beta}\,,$$

which closes the proof for (5.39). To control (5.40) we use the same decomposition as above, that is

$$\begin{aligned}&\Gamma^\alpha_{yx}(F\star G)^{<\beta}_x - (F\star G)^\alpha_y\\ &= \sum_{\nu_1<\beta-\mu_2} \Gamma^{\mu_1}_{yx}F^{\nu_1}_x\star\left(\Gamma^{\mu_2}_{yx}G^{<\beta-\nu_1}_x - G^{\mu_2}_y\right) - \left(\Gamma^{\mu_1}_{yx}F^{<\beta-\mu_2}_x - F^{\mu_1}_y\right)\star G^{\mu_2}_y\end{aligned}$$

and similarly for $\Gamma^\alpha_{yx}(\hat F\star\hat G)^{<\beta}_x - (\hat F\star\hat G)^\alpha_y$. Successive applications of the triangle inequality then yield with exactly the same estimates as above (5.40). □

We also have the following embedding result for the spaces $\mathcal{D}^{[\eta,\gamma]}$.

Lemma 5.3.8. *Given $\eta,\gamma,\eta',\gamma'\in\mathbb{R}\backslash A_{\mathbb{N}^d}$ such that $\eta\leqslant\gamma$, $\gamma\geqslant\gamma'$ and $\eta\geqslant\eta'$ we have the embedding $\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\mathcal{W})\subseteq\mathcal{D}^{[\eta',\gamma']}(\Omega^T;\mathcal{V}\backslash\mathcal{W})$, more precisely if α is the regularity of $\mathcal{V}\backslash\mathcal{W}$:*

$$\|F\|_{\mathcal{D}^{[\eta',\gamma']}(\Omega^T;\mathcal{V}\backslash\mathcal{W})} \lesssim (1+\|\Gamma\|_\eta)\cdot T^{\frac{\eta\wedge\alpha-\eta'}{\theta}\vee 0}\|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T);\mathcal{V}\backslash\mathcal{W})}\,. \tag{5.44}$$

Given a second model $(\hat\Pi,\hat\Gamma)$ on $\mathscr{T}$ we further have

$$\begin{aligned}\|F;\hat F\|_{\mathcal{D}^{[\eta',\gamma']}(\Omega^T;\mathcal{V}\backslash\mathcal{W},\Gamma,\hat\Gamma)} \lesssim T^{\frac{\eta\wedge\alpha-\eta'}{\theta}\vee 0}\big(&\|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T);\mathcal{V}\backslash\mathcal{W})}\,\|\Gamma-\hat\Gamma\|_\eta\\ &+(1+\|\hat\Gamma\|_\eta)\,\|F;\hat F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T);\mathcal{V}\backslash\mathcal{W},\Gamma,\hat\Gamma)}\big)\,.\end{aligned} \tag{5.45}$$

Proof. Using that for $\beta < \eta$ we have $\|F^{<\beta}\|_{\mathcal{D}^\eta(\Omega_t^T;\mathcal{V}\backslash\mathcal{W})} \lesssim (1+\|\Gamma\|_\eta)\|F^{<\eta}\|_{\mathcal{D}^\eta(\Omega_t^T;\mathcal{V}\backslash\mathcal{W})}$ we get (5.44) by splitting

$$\begin{aligned}\|F\|_{\mathcal{D}^{[\eta',\gamma']}(\Omega^T;\mathcal{V}\backslash\mathcal{W})} &\leqslant \sup_{t\in(0,T)} \sup_{\beta\in[\eta'\vee\alpha,\eta]\backslash A_{\mathbb{N}^d}} t^{\frac{\beta-\eta'}{\theta}} \|F^{<\beta}\|_{\mathcal{D}^\beta(\Omega_t^T;\mathcal{V}\backslash\mathcal{W})} \\ &+ \sup_{t\in(0,T)} \sup_{\beta\in[\eta,\gamma']\backslash A_{\mathbb{N}^d}} t^{\frac{\beta-\eta'}{\theta}} \|F^{<\beta}\|_{\mathcal{D}^\beta(\Omega_t^T;\mathcal{V}\backslash\mathcal{W})} \\ &\lesssim T^{\frac{\alpha-\eta'}{\theta}\vee 0}(1+\|\Gamma\|_\eta) \sup_{t\in(0,T)} \|F^{<\eta}\|_{\mathcal{D}^\eta(\Omega_t^T;\mathcal{V}\backslash\mathcal{W})} \\ &+ T^{\frac{\eta-\eta'}{\theta}} \|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\mathcal{W})} \\ &\lesssim (1+\|\Gamma\|_\eta)\cdot T^{\frac{\eta\wedge\alpha-\eta'}{\theta}\vee 0}\|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T);\mathcal{V}\backslash\mathcal{W})}\,.\end{aligned}$$

(5.45) follows by exactly the same arguments. □

We now consider again some product $\star$ on a regularity structure $\mathscr{T} = (A,\mathcal{T},G)$ which is equipped with a model (Π,Γ). Let $\mathcal{V}\subseteq\mathcal{T}$ be some *function like* sector in the sense of Definition 2.3.4. We assume that $\mathcal{V}$ is stable under $\star$, by what we mean that $\tau_1\star\tau_2\in\mathcal{V}$ holds for τ_1, $\tau_2\in\mathcal{V}$.

For $\gamma\in(0,\infty)\backslash A_{\mathbb{N}^d}$, $\eta\in(0,\gamma]\backslash A_{\mathbb{N}^d}$ such that $(\mathcal{V},\mathcal{V})$ is γ-regular in the sense of Definition 5.3.6 define the composition of a smooth $F\in C^\infty(\mathbb{R}^n,\mathbb{R})$, $n\geqslant 1$ with the vector $V=(V_1,\dots,V_n)\in(\mathcal{D}^{[\eta,\gamma]}(\mathcal{V}))^n$ as in [Hai14, Subsection 4.2] by

$$F(V) = \sum_k \frac{1}{k!}\partial^k F(V^{\mathbf{1}})\,(V-V^{\mathbf{1}}\mathbf{1})^{\star k}\,, \tag{5.46}$$

where we wrote $V^{\mathbf{1}} = (V_1^{\mathbf{1}},\dots,V_n^{\mathbf{1}})$ and use the notation $v^{\star k} := v_1^{\star k_1}\star\dots\star v_n^{\star k_n}$ for $v\in\mathcal{V}^n$ and $k\in\mathbb{N}^n$ (recall the definition of $\tau^{\mathbf{1}}$ as "the coefficient of τ before $\mathbf{1}$", compare page 46). Note that this sum is infinite (and therefore it is actually not contained in the direct sum $\bigoplus_{\alpha\in A}\mathcal{T}_\alpha$), but well-defined since for every homogeneity $\alpha\in A_{\mathcal{V}}$ only finitely many terms contribute to (5.46). We will work with the object $F^{<\gamma} = \sum_{\alpha\in A_{\mathcal{V}}:\,\alpha<\gamma}(F(V))^\alpha$ from now on. We have the following result, which can be seen as the analogue of [Hai14, Proposition 6.13] for the spaces $\mathcal{D}^{[\eta,\gamma]}$.

Lemma 5.3.9. *Let γ, η, $\mathcal{V}$ be as above, let (Π,Γ) be some model and let $F\in C^\infty(\mathbb{R}^n;\mathbb{R})$ be such that it has at most polynomially growing derivatives. We then have*

$$F^{<\gamma} : (\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V},\Gamma))^n \to \mathcal{D}^{[\eta-\kappa,\gamma]}(\Omega^T;\mathcal{V},\Gamma)$$

for any $\kappa>0$ and the norm of $F^{<\gamma}(V)$ for $V\in(\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V},\Gamma))^n$ is bounded by a polynomial in the norms of $V_1,\dots,V_n,\Gamma$. Given a second model $(\hat\Pi,\hat\Gamma)$ we further *have for* $V \in (\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V},\Gamma))^n$, $\hat{V} \in (\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V},\hat{\Gamma}))^n$ *the bound*

$$\|F^{<\gamma}(V); F^{<\gamma}(\hat{V})\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V},\Gamma,\hat{\Gamma})} \lesssim K \cdot (\|V;\hat{V}\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V},\Gamma,\hat{\Gamma})} + \|\Gamma - \hat{\Gamma}\|_\gamma)\,, \quad (5.47)$$

where we write $\|V;\hat{V}\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V},\hat{\Gamma},\Gamma)} = \sum_{i=1}^n \|V_i;\hat{V}_i\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V},\hat{\Gamma},\Gamma)}$ *and where* K *is some polynomial in the corresponding norms of* $V_1, \hat{V}_1, \ldots, V_n, \hat{V}_n, \Gamma$ *and* $\hat{\Gamma}$.

Remark 5.3.10. *If* F *has derivatives that grow faster than any polynomial we still have* $F : (\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}))^n \to \mathcal{D}^{[\eta-\kappa,\gamma]}(\Omega^T;\mathcal{V})$ *and* (5.47)*, but now* K *and* $F(V)$ *can be bounded uniformly for all* V, $\hat{V}$ *in any bounded set.*

Proof. We will use the notation K for a polynomial, that might change from line to line, in the norms of V, Γ (and $\hat{V}, \hat{\Gamma}$ in the last part of the proof). Fix some $\beta \in [\eta-\kappa,\gamma]$ and $\zeta \in (0,\eta)$ such that $\zeta < \min(A_{\mathcal{V}}\backslash\{0\}) \wedge 1$. We will use throughout the proof that due to Lemma 5.3.4 and Lemma 2.1.23 we have by our choice of ζ

$$\|V^{\mathbf{1}}\|_{\mathcal{C}_{\mathfrak{s}}^{\zeta}(\Omega^T)} \lesssim \|V^{<\zeta}\|_{\mathcal{D}^{\zeta}(\Omega^T;\mathcal{V})} \lesssim \|V\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V})} = K\,. \quad (5.48)$$

We have to estimate for $t \in (0,T)$ and $x, y \in \Omega_t^T$ with $\|y-x\|_{\mathfrak{s}} \leqslant 1$

$$\begin{aligned}\sum_{k\in\mathbb{N}^n} \frac{1}{k!}\partial^k F(V_y^{\mathbf{1}})\,((V_y - V_y^{\mathbf{1}})^{\star k})^\alpha - \Gamma_{yx}^\alpha \sum_{l\in\mathbb{N}^n} \frac{1}{l!}\partial^l F(V_x^{\mathbf{1}})\,((V_x - V_x^{\mathbf{1}}\mathbf{1})^{\star l})^{<\beta} \\ = \sum_{k\in\mathbb{N}^n:\,|k|\leqslant L} \frac{1}{k!}\partial^k F(V_y^{\mathbf{1}})\,((V_y - V_y^{\mathbf{1}}\mathbf{1})^{\star k})^\alpha \\ - \sum_{l\in\mathbb{N}^n:\,|l|\leqslant L} \frac{1}{l!}\,\partial^l F(V_x^{\mathbf{1}}) \sum_{0\leqslant m\leqslant l} \binom{l}{m} \Gamma_{yx}^\alpha((V_x)^{\star m})^{<\beta}\,(-V_x^{\mathbf{1}})^{(l-m)}\,.\end{aligned} \quad (5.49)$$

We can choose any $L \geqslant \beta/\zeta$ in the second line, since above β/ζ the terms under consideration vanish by our choice of ζ. We take

$$L = \lceil \beta/\zeta \rceil\,.$$

Let us emphasize that we really measure the size of the multi-indices k, l in isotropic scaling, that is $|k| = k_1 + \ldots + k_d$.

By Lemma 5.3.7 we have (since $\mathcal{V}$ is function like)

$$\|\Gamma_{yx}^\alpha((V_x)^{\star m})^{<\beta} - \left(V_y^{\star m}\right)^\alpha\|_{\mathcal{T}_\alpha} \lesssim K\, t^{\frac{\eta-\kappa-\beta}{\theta}}\, \|y-x\|_s^{\beta-\alpha}\,,$$

so that, due to (5.48), is enough to consider instead of (5.49)

$$\sum_{|k|\leqslant L}\frac{1}{k!}\partial^k F(V_y^{\mathbf{1}})((V_y - V_y^{\mathbf{1}}\mathbf{1})^{\star k})^{\alpha} - \sum_{|l|\leqslant L}\frac{1}{l!}\,\partial^l F(V_x^{\mathbf{1}})\sum_{m\leqslant l}\binom{l}{m}\left(V_y^{\star m}\right)^{\alpha}\,(-V_x^{\mathbf{1}})^{l-m}$$
$$= \sum_{|k|\leqslant L}\frac{1}{k!}\partial^k F(V_y^{\mathbf{1}})\,((V_y - V_y^{\mathbf{1}}\mathbf{1})^{\star k})^{\alpha}$$
$$- \sum_{|l|\leqslant L}\frac{1}{l!}\partial^l F(V_x^{\mathbf{1}})\sum_{m\leqslant l}\binom{l}{m}\left((V_y - V_y^{\mathbf{1}}\mathbf{1})^{\star(l-m)}\right)^{\alpha}\,(V_y^{\mathbf{1}} - V_x^{\mathbf{1}})^{m}\,,$$

where we used that the sum over m coincides with the projection of $(V_y - V_x^{\mathbf{1}}\mathbf{1})^{\star l} = \big(V_y - V_y^{\mathbf{1}}\mathbf{1} + (V_y^{\mathbf{1}} - V_x^{\mathbf{1}})\mathbf{1}\big)^{\star l}$ onto $\mathcal{V}_\alpha$. The last line can be reshaped as (via "$k = l-m$")

$$\sum_{|k|\leqslant L}\frac{1}{k!}((V_y - V_y^{\mathbf{1}}\mathbf{1})^{\star k})^{\alpha}\cdot\Big[\partial^k F(V_y^{\mathbf{1}}) - \sum_{m\colon |k+m|\leqslant L}\frac{1}{m!}\partial^{k+m}F(V_x^{\mathbf{1}})(V_y^{\mathbf{1}} - V_x^{\mathbf{1}})^{m}\Big]$$
$$= \sum_{|k|\leqslant L}\frac{1}{k!}((V_y - V_y^{\mathbf{1}}\mathbf{1})^{\star k})^{\alpha}\sum_{|r|=L+1-|k|}\frac{L+1-|k|}{r!}$$
$$\times\int_0^1 \mathrm{d}\zeta\,(1-\zeta)^{L-|k|}\partial^{k+r}F(V_{x+\zeta(y-x)}^{\mathbf{1}})\,(V_y^{\mathbf{1}} - V_x^{\mathbf{1}})^{r}\,,$$

where we used the (isotropic) multidimensional Taylor formula in the last step. The only k that contribute satisfy $|k|\leqslant \alpha/\zeta$, so that we can replace the last line by taking only

$$\sum_{|k|\leqslant \alpha/\zeta}\frac{1}{k!}((V_y^{\star k} - V_y^{\mathbf{1}}\mathbf{1})^{\star k})^{\alpha}\sum_{|r|=L+1-|k|}\frac{L+1-|k|}{r!}$$
$$\times\int_0^1 \mathrm{d}\zeta\,(1-\zeta)^{L-|k|}\partial^{k+r}F(V_{x+\zeta(y-x)}^{\mathbf{1}})\,(V_y^{\mathbf{1}} - V_x^{\mathbf{1}})^{r}\,. \tag{5.50}$$

Since further by the binomial theorem, (5.48) and once more Lemma 5.3.7

$$\big\|\big((V_y - V_y^{\mathbf{1}}\mathbf{1})^{\star k}\big)^{\alpha}\big\|_{\mathcal{T}_\alpha} \lesssim \sum_{m\leqslant k}|V_y^{\mathbf{1}}|^m\,\|(V_y^{\star(k-m)})^{\alpha}\|_{\mathcal{T}_\alpha}{}^{\alpha<\beta}$$
$$\lesssim K\sum_{0\leqslant m\leqslant k}\|V_y^{\star(k-m)}\|_{\mathcal{D}^\beta(\Omega_t^T)} \lesssim K\,t^{\frac{\eta-\kappa-\beta}{\theta}} \tag{5.51}$$

we can bound (5.50) due to (5.48) and our choice of $L\geqslant \beta/\zeta$ by

$$\sum_{|k|\leqslant\alpha/\zeta}Kt^{\frac{\eta-\kappa-\beta}{\theta}}\,\|y-x\|_{\mathfrak{s}}^{(L+1-|k|)\zeta} \leqslant \sum_{|k|\leqslant\alpha/\zeta}Kt^{\frac{\eta-\kappa-\beta}{\theta}}\,\|y-x\|_{\mathfrak{s}}^{(L-|k|)\zeta} \leqslant Kt^{\frac{\eta-\kappa-\beta}{\theta}}\|y-x\|_{\mathfrak{s}}^{\beta-\alpha}\,.$$

We now turn to the Lipschitz continuity of F. We split, once more, as above

$$F^\alpha(V_y) - \Gamma^\alpha_{yx} F(V_x)^{<\beta} = \sum_{|k| \leqslant \alpha/\zeta} A_k(V)\, B_k(V) + \sum_{|l| \leqslant L} C_l(V)\,,$$
$$F^\alpha(\hat V_y) - \Gamma^\alpha_{yx} F(\hat V_x)^{<\beta} = \sum_{|k| \leqslant \alpha/\zeta} A_k(\hat V)\, B_k(\hat V) + \sum_{|l| \leqslant L} \hat C_l(\hat V)\,,$$

where

$$A_k(V) = \frac{1}{k!}((V_y - V_y^{\mathbf{1}}\mathbf{1})^{\star k})^\alpha,$$
$$B_k(V) = \sum_{|r| = L+1-|k|} \frac{L+1-|k|}{r!} \int_0^1 \mathrm{d}\zeta\, (1-\zeta)^{L-|k|} \partial^{k+r} F(V^{\mathbf{1}}_{x+\zeta(y-x)})\, (V_y^{\mathbf{1}} - V_x^{\mathbf{1}})^r\,,$$
$$C_k(V) = \sum_{l \in \mathbb{N}^n \colon |l| \leqslant L} \frac{1}{l!}\, \partial^l F(V_x^{\mathbf{1}}) \sum_{0 \leqslant m \leqslant l} \binom{l}{m} \left[\Gamma^\alpha_{yx}((V_x)^{\star m})^{<\beta} - (V_y^{\star m})^\alpha\right] (-V_x^{\mathbf{1}})^{(l-m)}$$

(and similar for $\hat C_k(\hat V)$). For A_k, B_k, C_k one easily sees by the arguments above

$$\|A_k(V)\|_{\mathcal{T}^\alpha} \leqslant K t^{\frac{\eta-\kappa-\beta}{\theta}},\ \|A_k(V) - A_k(\hat V)\|_{\mathcal{T}_\alpha} \leqslant K t^{\frac{\eta-\kappa-\beta}{\theta}} \|V; \hat V\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T; \mathcal{V}, \Gamma, \hat\Gamma)}\,,$$
$$\|B_k(V)\|_{\mathcal{T}_\alpha} \leqslant K \|y-x\|_{\mathfrak{s}}^{\beta-\alpha}\,,$$
$$\|C_k(V) - \hat C_k(\hat V)\|_{\mathcal{T}_\alpha} \leqslant K t^{\frac{\eta-\kappa-\beta}{\theta}} \|y-x\|_{\mathfrak{s}}^{\beta-\alpha}(\|V; \hat V\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T; \mathcal{V}, \Gamma, \hat\Gamma)} + \|\Gamma; \hat\Gamma\|_\gamma)\,.$$

(5.47) follows from these estimates as soon as we can show $\|B_k(V) - B_k(\hat V)\|_{\mathcal{T}_\alpha} \leqslant K\|y-x\|_{\mathfrak{s}}^{\beta-\alpha} \|V; \hat V\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T; \mathcal{V}, \Gamma, \hat\Gamma)}$, which is true since we chose $L \geqslant \beta/\zeta$ so that for $r \in \mathbb{N}^n$, $|r| = L+1-|k|$ by the mean value theorem in $\mathbb{R}^n$

$$|(V_y^{\mathbf{1}} - V_x^{\mathbf{1}})^r - (\hat V_y^{\mathbf{1}} - \hat V_x^{\mathbf{1}})^r| \lesssim (|V_y^{\mathbf{1}} - V_x^{\mathbf{1}}|^{|r|-1} + |\hat V_y^{\mathbf{1}} - \hat V_x^{\mathbf{1}}|^{|r|-1}) \cdot \sup_z |V_z^{\mathbf{1}} - \hat V_z^{\mathbf{1}}|$$
$$= \sup_z |V_z^{\mathbf{1}} - \hat V_z^{\mathbf{1}}| \cdot (|V_y^{\mathbf{1}} - V_x^{\mathbf{1}}|^{L-|k|} + |\hat V_y^{\mathbf{1}} - \hat V_x^{\mathbf{1}}|^{L-|k|})$$
$$\lesssim \|V; \hat V\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T; \mathcal{V}, \Gamma, \hat\Gamma)}\, K\, \|y-x\|_{\mathfrak{s}}^{(L-|k|)\zeta} \leqslant \|V; \hat V\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T; \mathcal{V}, \Gamma, \hat\Gamma)}\, K \|y-x\|_{\mathfrak{s}}^{\beta-\alpha}\,.$$

We actually still have to bound the first terms in (2.46), (2.47) but this is once more an argument as in 5.51. □

We can also reformulate the reconstruction theorem, Theorem 2.3.19, for our spaces $\mathcal{D}^{[\eta,\gamma]}(\mathcal{V})$, which is a version of [Hai14, Proposition 6.9].

Lemma 5.3.11. *Let $\mathcal{V}$ be a sector of regularity $-\theta < \alpha \leqslant 0$. Let further $\gamma \in (0,\infty)\backslash A_{\mathbb{N}^d}$ and $\eta \in (-\theta,\gamma]\backslash A_{\mathbb{N}^d}$. Then, given some model (Π,Γ), there is for $F \in$*

$\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V},\Gamma)$ *a unique distribution* $\mathcal{R}F$ *which coincides within* Ω_t^T *with the unique distribution given by Theorem 2.3.19 and satisfies further for any* $\kappa > 0$

$$\|\mathcal{R}F\|_{\mathcal{C}_{\mathfrak{s}}^{\alpha\wedge\eta-\kappa}(\mathbb{R}^d;\mathbb{R})} \lesssim K_1 \cdot \|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;V)} \,. \tag{5.52}$$

where K_1 *is a polynomial in* $|\!|\!|(\Pi,\Gamma)|\!|\!|_\gamma$. *Given a second model* $(\hat\Pi,\hat\Gamma)$ *and a corresponding operator* $\hat{\mathcal{R}}$ *we further have*

$$\|\mathcal{R}F - \hat{\mathcal{R}}\hat F\|_{\mathcal{C}_{\mathfrak{s}}^{\alpha\wedge\eta-\kappa}(\mathbb{R}^d;\mathbb{R})} \lesssim K_2 \cdot \left(\|F;\hat F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V},\Gamma,\hat\Gamma)} + |\!|\!|(\Pi,\Gamma);(\hat\Pi,\hat\Gamma)|\!|\!|_\gamma\right). \tag{5.53}$$

where K_2 *is a polynomial in the norms of* F, $\hat F$, (Π,Γ) *and* $(\hat\Pi,\hat\Gamma)$.

Proof. Due to the continuous embedding $\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}) \subseteq \mathcal{D}^{\gamma,\eta-\kappa}(\Omega^T;\mathcal{V})$ for $\kappa > 0$ (page 125) this as consequence of [Hai14, Proposition 6.9]. □

5.3.2 A poor man's extension

We fix $\mathscr{T} = (A,\mathcal{T},G)$ and a scaling vector $\mathfrak{s} = (\theta,\mathfrak{s}')$ as in the last subsection. We further choose some model (Π,Γ) on $\mathscr{T}$. Suppose we are given $F : \Omega^T \to \mathcal{T}$. We then define an extension $\overline{F} : \mathbb{R}^d\backslash P^T \to \mathcal{T}$ by

$$\overline{F}_x = \Gamma_{x\overline{x}}F_{\overline{x}},\ x \in \mathbb{R}^d,$$

where $\overline{x} \in \Omega^T$ for $x = (x_1,x') \in \mathbb{R}^d\backslash P^T$ is defined by

$$\overline{x} = \begin{cases} x, & \text{for } x \in \Omega^T\,, \\ (-x_1 \wedge T/2, x'), & \text{for } x_1 < 0\,, \\ ((T-(x_1-T)) \vee T/2, x'), & \text{for } x_1 > T\,. \end{cases}$$

We then have the following basic, but useful lemma.

Lemma 5.3.12. *Let* $\mathcal{V}\backslash\mathcal{W}$ *be the complement of a sector* $\mathcal{W} \subseteq \mathcal{V}$ *within a sector* $\mathcal{V} \subseteq \mathcal{T}$. *For parameters* $\eta,\gamma \in \mathbb{R}\backslash A_{\mathbb{N}^d}$ *with* $\eta \leqslant \gamma$, *a modelled distribution* $F \in \mathcal{D}^{[\eta,\gamma]}(\Omega^T;\ \mathcal{V}\backslash\mathcal{W},\Gamma)$ *and points* $x,y \in \mathbb{R}^d\backslash P^T$ *of which at least one is contained in* Ω^T *we have for* $\beta \in [\eta,\gamma]\backslash A_{\mathbb{N}^d}$, $\alpha \in A_{\mathcal{V}\backslash\mathcal{W}}$, $\alpha < \beta$

$$\begin{aligned} &\left\| \left(\overline{F^{<\beta}}\right)_y^\alpha - \Gamma_{yx}^\alpha \left(\overline{F^{<\beta}}\right)_x \right\|_{\mathcal{T}_\alpha} \lesssim \|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\mathcal{W})}\,(1+\|\Gamma\|_\gamma) \\ &\times \left(\lceil x_1\rceil^{\frac{\eta-\beta}{\theta}} \vee \lceil y_1\rceil^{\frac{\eta-\beta}{\theta}}\right) \|y-x\|_{\mathfrak{s}}^{\beta-\alpha}\,, \end{aligned} \tag{5.54}$$

where $\lceil x \rceil = |x| \wedge T/2$. *Given a second model* $(\hat{\Pi}, \hat{\Gamma})$ *we also have for* $x,\, y \in \mathbb{R}^d \backslash P^T$

$$\begin{aligned}
&\left\| \left(\overline{F^{<\beta}}\right)_y^{\alpha} - \Gamma^{\alpha}_{yx}\left(\overline{F^{<\beta}}\right)_x - \left(\overline{\hat{F}^{<\beta}}\right)_y^{\alpha} + \hat{\Gamma}^{\alpha}_{yx}\left(\overline{\hat{F}^{<\beta}}\right)_x \right\|_{\mathcal{T}_\alpha} \\
&\lesssim \left(\lceil x_1 \rceil^{\frac{\eta-\beta}{\theta}} \vee \lceil y_1 \rceil^{\frac{\eta-\beta}{\theta}}\right) \|y-x\|_{\mathfrak{s}}^{\beta-\alpha} \\
&\times \Big[\|\Gamma\|_\gamma \, \|F;\hat{F}\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\mathcal{W},\Gamma;\hat{\Gamma})} + \big(\|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\mathcal{W},\Gamma)} \\
&+ \|\hat{F}\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\mathcal{W},\hat{\Gamma})}\big) \, \|\Gamma - \hat{\Gamma}\|_\gamma \Big] ,
\end{aligned} \tag{5.55}$$

where the extensions $\overline{(\dots)}$ *are taken in the corresponding model.*

Proof. We first show that for $a \in \Omega^T$, $b \in (\Omega^T)^c \backslash P^T$

$$\|a - \bar{b}\|_{\mathfrak{s}} < \|a-b\|_s,\ \|b-\bar{b}\|_{\mathfrak{s}} < 2\|a-b\|_{\mathfrak{s}} . \tag{5.56}$$

We show (5.56) by assuming $b_1 < 0$, the case $b_1 > T$ follows then by symmetry. The first inequality follows from

$$\|a-\bar{b}\|_{\mathfrak{s}} = |a_1 - \bar{b}_1|^{1/\mathfrak{s}_1} + \sum_{i=2}^{d} |a_i - b_i|^{1/\mathfrak{s}_i} < |a_1 - b_1|^{1/\mathfrak{s}_1} + \sum_{i=2}^{d} |a_i - b_i|^{1/\mathfrak{s}_i} = \|a-b\|_{\mathfrak{s}} ,$$

where we used $a_1,\, \bar{b}_1 > 0$, $\bar{b}_1 \leqslant |b_1|$ and $b_1 < 0$. For the second intequality note that $\|b - \bar{b}\|_s \leqslant 2|b_1|^{1/\mathfrak{s}_1}$, so that due to $a_1 > 0 > b_1$

$$\|a-b\|_{\mathfrak{s}} \geqslant |a_1 - b_1|^{1/\mathfrak{s}_1} > |b_1|^{1/\mathfrak{s}_1} \geqslant \frac{1}{2}\|b-\bar{b}\|_s .$$

Let us now get to (5.54). If $x,\, y \in \Omega^T$ there is nothing to prove. Assume that $x \in \Omega^T$, $y \in (\Omega^T)^c \backslash P^T$. We then have for $\alpha \in A_{\mathcal{V}\backslash\mathcal{W}}$

$$\begin{aligned}
\left(\overline{F^{<\beta}}\right)_y^{\alpha} - \Gamma^{\alpha}_{yx}\left(\overline{F^{<\beta}}\right)_x &= \Gamma^{\alpha}_{y\bar{y}} F^{<\beta}_{\bar{y}} - \Gamma^{\alpha}_{yx} F^{<\beta}_x = \Gamma^{\alpha}_{y\bar{y}}\left(F^{<\beta}_{\bar{y}} - \Gamma_{\bar{y}x} F^{<\beta}_x\right) \\
&= \sum_{\alpha' \in A_{\mathcal{V}\backslash\mathcal{W}}:\, \alpha \leqslant \alpha' < \beta} \Gamma^{\alpha}_{y\bar{y}}\left(F^{\alpha'}_{\bar{y}} - \Gamma^{\alpha'}_{\bar{y}x} F^{<\beta}_x\right) ,
\end{aligned}$$

so that the statement follows together with (5.56) from the fact that $x, \bar{y} \in \Omega^T$. For the case $(x \in \Omega^T)^c \backslash P^T$, $y \in \Omega^T$ we finally have

$$\left(\overline{F^{<\beta}}\right)_y^{\alpha} - \Gamma^{\alpha}_{yx}\left(\overline{F^{<\beta}}\right)_x = F^{\alpha}_y - \Gamma^{\alpha}_{yx}\Gamma_{x\bar{x}} F^{<\beta}_{\bar{x}} = F^{\alpha}_y - \Gamma^{\alpha}_{y\bar{x}} F^{<\beta}_{\bar{x}} ,$$

so that we conclude by using $\bar{x}, y \in \Omega^T$ and once more (5.56). The second estimate (5.55) follows by the same arguments. $\square$

Lemma 5.3.12 essentially tells us that that the "poor man's extension" $\overline{F}$ preserves the modelled distribution estimate for pairs of points, which contain at least one member contained in the set where F is (singular) modelled. This is enough to prove the following result, which will be of some importance for the Schauder estimates we derive in Section 6.2 below.

Proposition 5.3.13. *Assume that $\mathscr{T}$ satisfies Assumption 2.3.12. Let $\eta, \gamma \in \mathbb{R}\backslash A_{\mathbb{N}^d}$ with $\eta \leqslant \gamma$ and $F \in \mathcal{D}^{[\eta,\gamma]}(\Omega^T; \mathcal{T}\backslash\overline{\mathcal{T}})$ and suppose further that for some $\beta \in ([\eta,\gamma] \cap (0,\infty))\backslash A_{\mathbb{N}^d}$ with $\eta - \beta > -\mathfrak{s}_1$ and $t \in (0,T)$ we have an extension $\tilde{F}^{<\beta} \in \mathcal{D}^{\beta}(\mathbb{R}^d; \mathcal{T}\backslash\overline{\mathcal{T}})$ such that $\tilde{F}^{<\beta}|_{\Omega_t^T} = F^{<\beta}$ and $\|\tilde{F}^{<\beta}\|_{\mathcal{D}^{\beta}(\mathbb{R}^d;\mathcal{T}\backslash\overline{\mathcal{T}})} \lesssim t^{\frac{\eta-\beta}{\theta}} \|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{T}\backslash\overline{\mathcal{T}})}$. It then holds*

$$\|P(\overline{F^{<\beta}}, \Pi) - P(\tilde{F}^{<\beta}, \Pi)\|_{\mathcal{C}_{\mathfrak{s}}^{\beta}(\Omega_t^T;\mathbb{R})} \lesssim K_1 t^{\frac{\eta-\beta}{\theta}} \|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{T}\backslash\overline{\mathcal{T}})}\,, \tag{5.57}$$

where K_1 is a polynomial in $\|\!|(\Pi,\Gamma)|\!\|_{\gamma}$. Assume there is a second model $(\hat{\Pi},\hat{\Gamma})$, a function $\hat{F} \in \mathcal{D}^{[\eta,\gamma]}(\Omega_t^T, \mathcal{T}\backslash\overline{\mathcal{T}}, \hat{\Gamma})$ and a corresponding extension $\tilde{\hat{F}}^{<\beta} \in \mathcal{D}^{\beta}(\mathbb{R}^d; \mathcal{T}\backslash\overline{\mathcal{T}}, \hat{\Gamma})$ such that we have in addition $\|\tilde{F}^{<\beta}; \tilde{\hat{F}}^{<\beta}\|_{\mathcal{D}^{\beta}(\mathbb{R}^d;\mathcal{T}\backslash\overline{\mathcal{T}},\Gamma,\hat{\Gamma})} \lesssim t^{\frac{\eta-\beta}{\theta}} \|F;\hat{F}\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{T}\backslash\overline{\mathcal{T}},\Gamma,\hat{\Gamma})}$. In this case it holds

$$\begin{aligned}\|P(\overline{F^{<\beta}} - \tilde{F}^{<\beta}, \Pi) - P(\overline{\hat{F}^{<\beta}} - \tilde{\hat{F}}^{<\beta}, \hat{\Pi})\|_{\mathcal{C}_{\mathfrak{s}}^{\beta}(\Omega_t^T;\mathbb{R})} &\lesssim K_2 t^{\frac{\eta-\beta}{\theta}} \\ \times \left(\|\Pi - \hat{\Pi}\|_{\gamma} + \|F;\hat{F}\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{T}\backslash\overline{\mathcal{T}},\Gamma,\hat{\Gamma})}\right), &\end{aligned} \tag{5.58}$$

where K_2 is a polynomial in the corresponding norms of $F, \hat{F}, (\Pi,\Gamma)$ and $(\hat{\Pi},\hat{\Gamma})$ and where the extensions $\overline{(\dots)}$ are taken in the corresponding model.

Remark 5.3.14. *The reason why we are working with $F \in \mathcal{D}^{[\eta,\gamma]}(\Omega^T; \mathcal{T}\backslash\overline{\mathcal{T}}, \Gamma)$ instead of $F \in \mathcal{D}^{[\eta,\gamma]}(\Omega^T; \mathcal{V}\backslash\overline{\mathcal{T}}, \Gamma)$ for some sector $\mathcal{V}$ (as in the rest of this chapter) is that the extensions we construct in Theorem 5.3.16 below will typically not have the property that $\tilde{F}^{<\beta}$ takes values in $\mathcal{V}$ if this is true for F. However, if F only takes values in $\mathcal{V} \subseteq \mathcal{T}$ we have $\|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega_t^T,\mathcal{T}\backslash\overline{\mathcal{T}},\Gamma)} = \|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega_t^T,\mathcal{V}\backslash\overline{\mathcal{T}},\Gamma)}$ so that this is in fact just a notational issue.*

Proof. Note first that the function $\Delta F := \tilde{F}^{<\beta} - \overline{F^{<\beta}}$ satisfies for $x \in \Omega_t^T$, $y \in \mathbb{R}^d \backslash P^T$ and $\alpha \in A_{\mathcal{T}\backslash\overline{\mathcal{T}}}$, $\alpha < \beta$

$$\begin{aligned}\|\Gamma_{xy}^{\alpha}\Delta F_y\|_{\mathcal{T}_\alpha} &= \left\|\Gamma_{xy}^{\alpha}\tilde{F}_y^{<\beta} - F_x^{\alpha} + F_x^{\alpha} - \Gamma_{xy}^{\alpha}\left(\overline{F^{<\beta}}\right)_y\right\|_{\mathcal{T}_\alpha} &(5.59)\\ &= \left\|\Gamma_{xy}^{\alpha}\tilde{F}_y^{<\beta} - (\tilde{F}_x^{<\beta})^{\alpha} + \left(\overline{F^{<\beta}}\right)_x^{\alpha} - \Gamma_{xy}^{\alpha}\left(\overline{F^{<\beta}}\right)_y\right\|_{\mathcal{T}_\alpha} &\\ &\lesssim (t^{\frac{\eta-\beta}{\theta}} + \lceil y_1 \rceil^{\frac{\eta-\beta}{\theta}})\|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{T}\backslash\overline{\mathcal{T}})}\|y-x\|_{\mathfrak{s}}^{\beta-\alpha}\,, &(5.60)\end{aligned}$$

due to Lemma 5.3.12, where $\lceil\cdot\rceil$ is defined as in Lemma 5.3.12. Similar with $\Delta\hat{F} := \overline{\hat{F}^{<\beta}} - \tilde{\hat{F}}^{<\beta}$, $x \in \Omega_t^T$

$$\|\Gamma^\alpha_{xy}\Delta F_y - \hat{\Gamma}^\alpha_{xy}\Delta\hat{F}_y\|_{\mathcal{T}_\alpha} \lesssim (t^{\frac{\eta-\beta}{\theta}} + \lceil y_1\rceil^{\frac{\eta-\beta}{\theta}})\|F;\hat{F}\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{T}\backslash\overline{\mathcal{T}},\Gamma,\hat{\Gamma})}\|y-x\|_{\mathfrak{s}}^{\beta-\alpha}\,. \tag{5.61}$$

We will need the following objects for $j \geqslant -1, l \in \mathbb{N}^d$, $\alpha \in A_{\mathcal{T}\backslash\overline{\mathcal{T}}}$, $x' \in \Omega_t^T$

$$f_{x'}^{(l),\alpha} := \int \mathrm{d}u\, \partial^l \Psi^{<j-1}_{x'-u}\Gamma^\alpha_{x'u}\Delta F_u\,, \qquad \hat{f}_{x'}^{(l),\alpha} := \int \mathrm{d}u\, \partial^l \Psi^{<j-1}_{x'-u}\hat{\Gamma}^\alpha_{x'u}\Delta\hat{F}_u\,.$$

With (5.60) and (5.61) and Lemma 5.3.15 below (together with Lemma 2.1.14) we obtain the bounds

$$\|f_{x'}^{(l),\alpha}\|_{\mathcal{T}_\alpha} \lesssim K_1\, 2^{-j(\beta-|l|_{\mathfrak{s}}-\alpha)}t^{\frac{\eta-\beta}{\theta}}\|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{T}\backslash\overline{\mathcal{T}})}, \tag{5.62}$$

$$\|f_{x'}^{(l),\alpha} - \hat{f}_{x'}^{(l),\alpha}\|_{\mathcal{T}_\alpha} \lesssim K_2\, 2^{-j(\beta-|l|_{\mathfrak{s}}-\alpha)}t^{\frac{\eta-\beta}{\theta}}\|F;\hat{F}\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{T}\backslash\overline{\mathcal{T}})}\,. \tag{5.63}$$

We first show (5.57). In view of Lemma 2.1.23 we have to estimate for $x \in \Omega_t^T$, $x+h \in \Omega_t^T$ with $0 < \|h\|_s \leqslant 1$ and $\overline{k} \in \mathbb{N}^d_{<\beta}$

$$\partial^{\overline{k}} R_{x;h}^{\beta-|\overline{k}|_{\mathfrak{s}}} P(\Delta F, \Pi) = \sum_{0\leqslant\overline{l}\leqslant\overline{k}}\binom{\overline{k}}{\overline{l}}\sum_{j>0}\iint \mathrm{d}u\, \Pi_u\Delta F_u(\mathrm{d}v) R_{x;h}^{\beta-|\overline{k}|_{\mathfrak{s}}}\left(\partial^{\overline{l}}\Psi^{<j-1}_{\cdot-u}\partial^{\overline{k}-\overline{l}}\Psi^j_{\cdot-v}\right) \tag{5.64}$$

$$=: \sum_{0\leqslant\overline{l}\leqslant\overline{k}}\binom{\overline{k}}{\overline{l}}\sum_{j>0} R_j^{\overline{l}}(x)\,. \tag{5.65}$$

We will fix a $j' \geqslant -1$ such that $2^{-j'-1} < \|h\|_{\mathfrak{s}} \leqslant 2^{-j'}$ and estimate the terms $R_j^{\overline{l}}$ in the expansion(5.65) separately for $j \leqslant j'$ and $j > j'$. In the low frequency case, that is $j \leqslant j'$, we apply the anisotropic Taylor formula from Lemma 2.1.20 and Leibniz's rule to reshape $R_j^{\overline{l}}$ as

$$R_j^{\overline{l}}(x) = \sum_{\substack{k\,\in\,\mathbb{N}^d_{>\beta-|\overline{k}|_{\mathfrak{s}}}\\ 0\,\leqslant\, l\,\leqslant\, k}} c_{k,l}\, h^k\int_0^1 \mathrm{d}\zeta(1-\zeta)^{k_{\mathfrak{m}(k)}-1}\sum_{\alpha\in A_{\mathcal{T}\backslash\overline{\mathcal{T}}}}\Pi_{x(\zeta)}f_{x(\zeta)}^{(\overline{l}+l),\alpha}\left(\partial^{\overline{k}-\overline{l}+k-l}\Psi^j_{x(\zeta)-\cdot}\right)\,,$$

where $c_{k,l}$ denote constants, $x(\zeta) := x + v_\zeta^k(h) \in \Omega_t^T$ and where we used that polynomial entries vanish to restrict ourselves to $\alpha \in A_{\mathcal{T}\backslash\overline{\mathcal{T}}}$. With (5.62) we obtain the estimate

$$\begin{aligned}\sum_{j\leqslant j'}|R_j^{\overline{l}}(x)| &\lesssim \sum_{j\leqslant j'}\sum_{k\in\mathbb{N}^d_{>\beta-|\overline{k}|_{\mathfrak{s}}}}\|h\|_{\mathfrak{s}}^{|k|_{\mathfrak{s}}}2^{j(|k|_{\mathfrak{s}}-(\beta-|\overline{k}|_{\mathfrak{s}}))}t^{\frac{\eta-\beta}{\theta}}\,\|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{T}\backslash\overline{\mathcal{T}})})\\ &\lesssim \|h\|_{\mathfrak{s}}^{\beta-|\overline{k}|_{\mathfrak{s}}}t^{\frac{\eta-\beta}{\theta}}\,\|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{T}\backslash\overline{\mathcal{T}})}\,.\end{aligned}$$

It remains to bound $\sum_{j>j'} R^{\bar{l}}_j$. To this end we spell out

$$R^{\beta-|\bar{k}|_{\mathfrak{s}}}_{x;h}\left(\partial^{\bar{l}}\Psi^{<j-1}_{\cdot-u}\partial^{\bar{k}-\bar{l}}\Psi^j_{\cdot-v}\right) = \partial^{\bar{l}}\Psi^{<j-1}_{x+h-u}\partial^{\bar{k}-\bar{l}}\Psi^j_{x+h-u} - T^{\beta-|\bar{k}|_{\mathfrak{s}}}_{x;h}\left(\partial^{\bar{l}}\Psi^{<j-1}_{\cdot-u}\partial^{\bar{k}-\bar{l}}\Psi^j_{\cdot-v}\right)$$

and estimate all the terms separately, we have (with some new constants $c_{l,k}$)

$$\begin{aligned} R^{\bar{l}}_j =& \int \mathrm{d}u\, \partial^{\bar{l}}\Psi^{<j-1}_{x+h-u}\Pi_u \Delta F_u(\partial^{\bar{k}-\bar{l}}\Psi^j_{x+h-\cdot}) \\ &- \sum_{\substack{k\,\in\,\mathbb{N}^d_{<\beta-|\bar{k}|_{\mathfrak{s}}}\\ 0\leqslant l\leqslant k}} c_{l,k}\, h^k \int \mathrm{d}u\, \partial^{\bar{l}+l}\Psi^{<j-1}_{x-u}\, \Pi_u \Delta F_u(\partial^{\bar{k}-\bar{l}+k-l}\Psi^j_{x-\cdot}) \\ &= \Pi_{x+h} f^{(\bar{l}),\alpha}_{x+h}(\partial^{\bar{k}-\bar{l}}\Psi^j_{x+h-\cdot}) - \sum_{\substack{k\,\in\,\mathbb{N}^d_{<\beta-|\bar{k}|_{\mathfrak{s}}}\\ 0\leqslant l\leqslant k}} c_{l,k}\, h^k\, \Pi_x f^{(\bar{l}+l),\alpha}_x(\partial^{\bar{k}-\bar{l}+k-l}\Psi^j_{x-\cdot})\,. \end{aligned} \tag{5.66}$$

This can expression can be bounded using once more (5.62).

$$\begin{aligned} \sum_{j>j'} |R'_j| &\lesssim \sum_{j>j'} \big(2^{-j(\beta-|\bar{k}|_{\mathfrak{s}})} + \sum_{k\in\mathbb{N}^d_{<\beta-|\bar{k}|_{\mathfrak{s}}}} 2^{-j(\beta-|\bar{k}|_{\mathfrak{s}}-|k|_{\mathfrak{s}})}\|h\|^{|k|_{\mathfrak{s}}}_{\mathfrak{s}}\big)\cdot t^{\frac{\eta-\beta}{\theta}}\, \|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{T}\setminus\overline{\mathcal{T}})} \\ &\lesssim \|h\|^{\beta-|\bar{k}|_{\mathfrak{s}}} t^{\frac{\eta-\beta}{\theta}}\, \|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{T}\setminus\overline{\mathcal{T}})}\,. \end{aligned}$$

To finish the proof of (5.54) it remains to check that $P(\Delta F, \Pi)$ is actually a function on Ω^T_t whose derivatives up to order β are bounded by the right hand side of the claim, but this is just once more an estimate like the one for (5.66).

For (5.55) one proceeds precisely as above but uses also (5.63). □

Lemma 5.3.15. *Given some $\phi \in \mathcal{S}(\mathbb{R}^d)$, $t \in (0,T)$ and $x \in \Omega^T_t$ we have for $\nu \geqslant 0$ and $0 \leqslant \mu < \mathfrak{s}_1$*

$$\int_{\mathbb{R}^d} \mathrm{d}u\, |2^{j|\mathfrak{s}|}\phi(2^{j\mathfrak{s}}(x-u))|\, \lceil u_1 \rceil^{-\frac{\mu}{\mathfrak{s}_1}}\, \|x-u\|^\nu_{\mathfrak{s}} \lesssim_\phi 2^{-j\nu} t^{-\mu/\mathfrak{s}_1}\,,$$

uniformly in $j \geqslant -1$, where $\lceil\cdot\rceil$ is defined as in Lemma 5.3.12.

Proof. First note that we can restrict the integral in the claim to

$$\int_{u:\,|u_1|\leqslant t/2} \mathrm{d}u\, |2^{j|\mathfrak{s}|}\phi(2^j(x-u))|\, |u_1|^{-\frac{\mu}{\mathfrak{s}_1}}\, \|x-u\|^\nu_{\mathfrak{s}}$$

since on the complementary integration set we can estimate $\lceil u_1 \rceil^{-\frac{\mu}{\theta}}$ and then apply the standard estimates. Considering $\phi\|\cdot\|^\nu_{\mathfrak{s}}$ and using that ϕ is Schwartz we can reduce the problem to the task to estimate

$$\int_{|u_1|\leqslant t/2} \mathrm{d}u_1\, 2^{j\mathfrak{s}_1}\tilde{\phi}(2^{j\mathfrak{s}_1}(x_1-u_1))|u_1|^{-\mu/\theta} \lesssim t^{-\mu/\mathfrak{s}_1}\,, \tag{5.67}$$

where $\tilde{\phi}$ is a positive, possibly non-smooth function on $\mathbb{R}$ that can be bounded by the inverse of any polynomial. If $t < 2^{-j\mathfrak{s}_1}$ we simply estimate $\tilde{\phi} \lesssim 1$ so that the right hand side of (5.67) is up to a constant less than $2^{j\mathfrak{s}_1}\int_{|u_1|\leqslant t/2}|u_1|^{-\mu/\mathfrak{s}_1} \lesssim 2^{j\mathfrak{s}_1}t^{\frac{\mathfrak{s}_1-\mu}{\mathfrak{s}_1}} \leqslant t^{-\mu/\mathfrak{s}_1}$. For $2^{-j\mathfrak{s}_1} \leqslant t$ we use $\tilde{\phi}(y) \lesssim |y|_{\mathfrak{s}}^{-(\mu+\mathfrak{s}_1)/\mathfrak{s}_1}$ and $|x_1 - u_1| \geqslant t/2$, so that the left hand side of (5.67) can then be bounded by

$$\frac{2^{-j\mu}}{t^{(\mathfrak{s}_1+\mu)/\mathfrak{s}_1}}\int_{u_1\leqslant t/2}\mathrm{d}u_1\,|u_1|^{-\mu/\mathfrak{s}_1} \lesssim \frac{2^{-j\mu}}{t^{(\mathfrak{s}_1+\mu)/\mathfrak{s}_1}}\,t^{\frac{\mathfrak{s}_1-\mu}{\mathfrak{s}_1}} \leqslant t^{-\mu/\mathfrak{s}_1}\,.$$

□

5.3.3 A Whitney extension for modelled distributions

In this subsection we present the solution to a more delicate extension problem. Suppose we are given a regularity structure $\mathscr{T} = (A, \mathcal{T}, G)$ with a scaling vector $\mathfrak{s} \in [1,\infty)^d$ equipped with a model (Π, Γ) and satisfying Assumption 2.3.12. Take a sector $\mathcal{V} \subseteq \mathcal{T}$ and some $\gamma \in \mathbb{R}$. Suppose further we are given a non-empty set $\Omega \subseteq \mathbb{R}^d$ and some $F \in \mathcal{D}^\gamma(\Omega; \mathcal{V}, \Gamma)$. Can we find an extension $\mathcal{E}_\Omega F : \mathbb{R}^d \to \mathcal{T}$ such that $\|\mathcal{E}_\Omega F\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{T})} \lesssim \|F\|_{\mathcal{D}^\gamma(\Omega;\mathcal{V})}$ with the involved constant being independent of Ω?

The answer is yes, provided that $(\mathcal{V}, \overline{\mathcal{T}})$ is γ-regular under some product $\star$ on $\mathcal{T}$. We are grateful to M. Hairer for his suggestion to generalize the ideas from [Whi34, Ste70], which we will do from now on.

Since for $\alpha \in A_{\mathcal{V}}$ the map $\mathbb{R}^{2d} \ni (x,y) \mapsto \Gamma^\alpha_{yx}\tau$, $\tau \in \mathcal{T}$ and $F^\alpha : \Omega \to \mathcal{V}_\alpha$ are Hölder continuous with some (possibly small) exponent, we can extend F first to $\overline{\Omega}$ such that $F \in \mathcal{D}^\gamma(\overline{\Omega}; \mathcal{V})$ with $\|F\|_{\mathcal{D}^\gamma(\overline{\Omega};\mathcal{V})} = \|F\|_{\mathcal{D}^\gamma(\Omega;\mathcal{V})}$. We then decompose similar as in [Ste70] $\overline{\Omega}^c$ in a countable family of closed sets $(Q^n)_{n\in\mathbb{N}} \subseteq \overline{\Omega}^c$, which are in the form of $x + 2^{-m\mathfrak{s}}(1+\varepsilon)[0,1]^d$ for some varying $x \in \mathbb{R}^d$, $m \in \mathbb{N}$ and some fixed $\varepsilon > 0$, such that the following properties are satisfied for $n \in \mathbb{N}$ [2]

$$\mathrm{diam}_{\mathfrak{s}}Q^n \approx \mathrm{dist}_{\mathfrak{s}}(Q^n, \overline{\Omega})\,, \qquad |\{n' \in \mathbb{N}\,|\,Q_{n'} \cap Q_n \neq \varnothing\}| \leqslant C\,, \qquad \overline{\Omega}^c = \bigcup_{n'\in\mathbb{N}} Q^{n'}\,,$$

where $C = C(d) > 0$ is some fixed number, *independent of* Ω, and where $\mathrm{diam}_{\mathfrak{s}}$ and $\mathrm{dist}_{\mathfrak{s}}$ are defined as in Section 2.1. We will denote by p^n some arbitrary chosen point $p^n \in \overline{\Omega}$ such that $\mathrm{dist}_{\mathfrak{s}}(Q^n, \overline{\Omega}) = \mathrm{dist}_{\mathfrak{s}}(Q^n, \Omega) = \mathrm{dist}_{\mathfrak{s}}(Q^n, p^n)$ and we will use the set $N(\Omega) := \{n \in \mathbb{N}\,|\mathrm{dist}_{\mathfrak{s}}(Q^n, \overline{\Omega}) \leqslant 1\}$.

[2]In [Ste70] the author uses the isotropic scaling $\mathfrak{s} = (1,\ldots,1)$. However the construction generalizes readily to the anisotropic case.

There is an adapted partition of unity $\varphi^n \in C_c^\infty(\mathbb{R}^d; [0,1])$, $\operatorname{supp} \varphi^n \subseteq Q^n$ such that for every $x \in \overline{\Omega}^c$, $k \in \mathbb{N}^d$

$$\sum_{n \in \mathbb{N}} \varphi^n(x) = 1 , \qquad |\partial^k \varphi^n(x)| \lesssim (\operatorname{diam}_s Q^n)^{-|k|_\mathfrak{s}} \approx (\operatorname{dist}_\mathfrak{s}(Q^n, \Omega))^{-|k|_\mathfrak{s}} .$$

We denote by $\Phi^n := \sum_{k \in \mathbb{N}^d} \frac{1}{k!} \partial^k \varphi^n X^k$ the lift of φ^n to $\overline{\mathcal{T}}$, so that

$$\|\Phi^n\|_{\mathcal{D}^{\gamma'}(\mathbb{R}^d; \overline{\mathcal{T}})} \lesssim (\operatorname{dist}_\mathfrak{s}(Q^n, \Omega))^{-\gamma'} \tag{5.68}$$

for $\gamma' > 0$. We then finally define

$$\mathcal{E}_\Omega F(x) := \mathcal{E}_\Omega^\Gamma F(x) := \begin{cases} F_x^{<\gamma} & , x \in \overline{\Omega} \\ \sum_{n \in N(\Omega)} (\Gamma_{xp^n} F_{p^n} \star \Phi_x^n)^{<\gamma} & , x \in \overline{\Omega}^c \end{cases} . \tag{5.69}$$

The upper index "Γ" will be omitted if there is no risk of confusion. Note that by the choice of φ^n/Φ^n the sum on the right hand side for $x \in \overline{\Omega}^c$ is finite in every subspace $\mathcal{T}_\alpha$ as the sum actually only runs over the finite set

$$N_x(\Omega) := \{n \in N(\Omega) \,|\, x \in Q^n\}$$

since $\operatorname{supp} \Phi^n \subseteq Q^n$. In fact, by our choice of $(Q^n)_{n \in \mathbb{N}}$ we have $|N_x(\Omega)| \leqslant C$ with $C = C(d) > 0$ as above.

We will sometimes write $\mathcal{E}_\Omega F$ even if F is defined on some bigger set $\Omega' \supseteq \Omega$, by what we then mean $\mathcal{E}_\Omega(F|_\Omega)$. As usual, we have an abbreviated notation for the components in $\mathcal{T}_\alpha$ for $\alpha \in A$, namely

$$\mathcal{E}_\Omega^\alpha F := (\mathcal{E}_\Omega F)^\alpha$$

and define the notation $\mathcal{E}_\Omega^{<\beta} = \sum_{\alpha < \beta} \mathcal{E}_\Omega^\alpha$ for $\beta \in \mathbb{R}$. Our main result in this subsection is the following.

Theorem 5.3.16. *$\mathcal{E}_\Omega$ is a continuous linear operator from $\mathcal{D}^\gamma(\Omega; \mathcal{V})$ to $\mathcal{D}^\gamma(\mathbb{R}^d; \mathcal{T})$, .i.e.*

$$\|\mathcal{E}_\Omega F\|_{\mathcal{D}^\gamma(\mathbb{R}^d; \mathcal{T})} \lesssim K_1 \cdot \|F\|_{\mathcal{D}^\gamma(\Omega; \mathcal{V})} , \tag{5.70}$$

where the involved constant is independent of Ω.and where K_1 is some polynomial in $\|\Gamma\|_\gamma$. Given a second model $(\hat{\Pi}, \hat{\Gamma})$ we have in addition

$$\|\mathcal{E}_\Omega^\Gamma F; \mathcal{E}_\Omega^{\hat{\Gamma}} \hat{F}\|_{\mathcal{D}^\gamma(\mathbb{R}^d; \mathcal{T}, \Gamma, \hat{\Gamma})} \lesssim K_2 \cdot (\|F; \hat{F}\|_{\mathcal{D}^\gamma(\Omega; \mathcal{V}, \Gamma, \hat{\Gamma})} + \|\Gamma - \hat{\Gamma}\|_\gamma) , \tag{5.71}$$

where K_2 is a polynomial in the norms of F, $\hat{F}$, Γ and $\hat{\Gamma}$.

Proof. We first show (5.70), since this step contains all the necessary ideas. We will write $\mathcal{E}F$ instead of $\mathcal{E}_\Omega F$ and assume without loss of generality that $\|F\|_{\mathcal{D}^\gamma(\Omega;\mathcal{V})} \leqslant 1$. For $x \in \mathbb{R}^d$ we use the notation $\mathcal{E}_x F := (\mathcal{E}F)_x$.

Let us first prove $\|\mathcal{E}_x^\alpha F\|_{\mathcal{T}_\alpha} \lesssim 1$ for $\alpha \in A$, $\alpha < \gamma$ and $x \in \mathbb{R}^d$. For $x \in \overline{\Omega}$ this is obvious. For $x \in \overline{\Omega}^c$ with $\mathrm{dist}_\mathfrak{s}(x, \Omega) > 1$ this follows from a rather short estimate:

$$\begin{aligned}
\|\mathcal{E}_x^\alpha F\|_{\mathcal{T}_\alpha} &= \Big\| \sum_{n \in N_x(\Omega)} \sum_{\mu_1 + \mu_2 = \alpha} \Gamma^{\mu_1}_{xp^n} F_{p^n} * (\Phi^n_x)^{\mu_2} \Big\|_{\mathcal{T}_\alpha} \\
&\overset{(5.68)}{\lesssim} \sum_{n \in N_x(\Omega)} \sum_{\mu_1 + \mu_2 = \alpha} \|\Gamma^{\mu_1}_{xp^n} F_{p^n}\|_{\mathcal{T}_{\mu_1}} \cdot \mathrm{dist}_\mathfrak{s}(Q^n, \Omega)^{-\mu_2} \lesssim 1\,,
\end{aligned}$$

where we used in the last step that $\|\Gamma^{\mu_1}_{xp^n} F_{p^n}\|_{\mathcal{T}_{\mu_1}} \lesssim 1$ by definition of p^n and $N_x(\Omega)$ and that $\mathrm{dist}_\mathfrak{s}(Q^n, \Omega) \gtrsim \mathrm{dist}_\mathfrak{s}(x, \Omega) \geqslant 1$ by definition of Q^n.

Let therefore $x \in \overline{\Omega}^c$ be such that $\mathrm{dist}_\mathfrak{s}(x, \Omega) \leqslant 1$. Pick a $x' \in \overline{\Omega}$ such that $\mathrm{dist}_\mathfrak{s}(x, \overline{\Omega}) = \mathrm{dist}_\mathfrak{s}(x, \Omega) = \|x - x'\|_\mathfrak{s}$. We can then estimate

$$\begin{aligned}
\|\mathcal{E}_x^\alpha F\|_{\mathcal{T}_\alpha} &\leqslant \sum_{n \in N_x(\Omega)} \sum_{\mu_1 + \mu_2 = \alpha} \|\Gamma^{\mu_1}_{xp^n} (F_{p^n} - \Gamma_{p^n x'} F_{x'}) \star (\Phi^n_x)^{\mu_2}\|_{\mathcal{T}_\alpha} + \|\Gamma^\alpha_{xx'} F_{x'}\|_{\mathcal{T}_\alpha} \\
&\lesssim \sum_{n \in N_x(\Omega)} \sum_{\mu_1 + \mu_2 = \alpha} \sum_{\mu_1 \leqslant \beta < \gamma} \|p^n - x\|_\mathfrak{s}^{\beta - \mu_1} \, \|x' - p^n\|_\mathfrak{s}^{\gamma - \beta} \, \mathrm{dist}_\mathfrak{s}(x, \Omega)^{-\mu_2} + 1 \\
&\lesssim \mathrm{dist}_\mathfrak{s}(x, \Omega)^{\gamma - \alpha} + 1 \lesssim 1\,,
\end{aligned} \tag{5.72}$$

where we used in the first step $\sum_{n \in N_x(\Omega)} \Phi^n(x) = \sum_{n \in \mathbb{N}} \Phi^n(x) = \mathbf{1}$ due to $0 < \mathrm{dist}_\mathfrak{s}(x, \Omega) \leqslant 1$ and in the last step $\|x' - p^n\|_\mathfrak{s} \leqslant \|x' - x\|_\mathfrak{s} + \|x - p^n\|_\mathfrak{s} \lesssim \mathrm{dist}_\mathfrak{s}(x, \Omega)$ since by our choice of $(Q^n)_{n \in \mathbb{N}}$ we have

$$\|x - p^n\|_\mathfrak{s} \lesssim \mathrm{dist}_\mathfrak{s}(x, \Omega)$$

for $n \in N_x(\Omega)$.

We now prove for $x, y \in \mathbb{R}^d$ the estimate $\|\mathcal{E}_y^\alpha F - \Gamma^\alpha_{yx} \mathcal{E}_x F\|_{\mathcal{T}^\alpha} \lesssim \|y - x\|_\mathfrak{s}^{\gamma - \alpha}$ for $\alpha \in A$, $\alpha < \gamma$. Since for $x, y \in \overline{\Omega}$ this is clear, we are left with the following four cases:

1. $y \in \overline{\Omega}^c$, $x \in \overline{\Omega}$,
2. $y \in \overline{\Omega}$, $x \in \overline{\Omega}^c$,
3. $x, y \in \overline{\Omega}^c$ with $\mathrm{dist}_\mathfrak{s}([x, y], \overline{\Omega}) < 2\|x - y\|_\mathfrak{s}$,
4. $x, y \in \overline{\Omega}^c$ with $\|x - y\|_\mathfrak{s} \leqslant \frac{1}{2}\mathrm{dist}_\mathfrak{s}([x, y], \overline{\Omega})$,

where $[x,y] = x + [0,1]\,(y-x)$. Note that we allow in case 3 the distance $\mathrm{dist}_{\mathfrak{s}}([x,y],\overline{\Omega})$ to be 0, so that $[x,y]$ is allowed to intersect with $\overline{\Omega}$ and there is no implicit assumption on convexity of $\overline{\Omega}^c$ used here.

For case 1, note that is enough to consider y with $\mathrm{dist}_{\mathfrak{s}}(y,\Omega) \leqslant 1$, since otherwise we can apply the boundedness of components of F within $\overline{\Omega}$ and the definition of a model. Assume first that $x \in \overline{\Omega}$ is such that $\mathrm{dist}_{\mathfrak{s}}(y,\Omega) = \|y-x\|_{\mathfrak{s}}$, we then rewrite for $\alpha < \gamma$

$$\begin{aligned}\mathcal{E}^\alpha_y F - \Gamma^\alpha_{yx}\mathcal{E}_x F &= \sum_{n\in N_y(\Omega)} \sum_{\mu_1+\mu_2=\alpha} \Gamma^{\mu_1}_{yp^n} F_{p^n} \star (\Phi^n_y)^{\mu_2} - \Gamma^\alpha_{yx} F_x \\ &= \sum_{n\in N_y(\Omega)} \sum_{\mu_1+\mu_2=\alpha} (\Gamma^{\mu_1}_{yp^n} F_{p^n} - \Gamma^{\mu_1}_{yx} F_x) \star (\Phi^n_y)^{\mu_2}\,, \end{aligned} \tag{5.73}$$

where we used $\sum_{n\in N_y(\Omega)} \Phi^n(y) = \mathbf{1}$ in the second step. Since we have $(\Gamma^{\mu_1}_{yp^n} F_{p^n} - \Gamma^{\mu_1}_{yx} F_x) = \sum_{\mu_1\leqslant\nu<\gamma} \Gamma^{\mu_1}_{yp^n}(F^\nu_{p^n} - \Gamma^\nu_{p^n x} F_x)$ we can bound this expression by

$$\sum_{n\in N_y(\Omega)} \sum_{\mu_1+\mu_2=\alpha} \sum_{\mu_1\leqslant\nu<\gamma} \|y-p^n\|_{\mathfrak{s}}^{\nu-\mu_1} \|p^n - x\|_{\mathfrak{s}}^{\gamma-\nu} (\mathrm{diam}_{\mathfrak{s}} Q^n)^{-\mu_2} \lesssim \|y-x\|_{\mathfrak{s}}^{\gamma-\alpha}\,,$$

where we used that $\|y-p^n\|_{\mathfrak{s}} \lesssim \mathrm{dist}_{\mathfrak{s}}(y,\Omega) = \|y-x\|_{\mathfrak{s}}$, $\|x-p^n\|_{\mathfrak{s}} \leqslant \|x-y\|_{\mathfrak{s}} + \|y-p^n\|_{\mathfrak{s}} \lesssim \|y-x\|_{\mathfrak{s}}$ and $\|y-x\|_{\mathfrak{s}} = \mathrm{dist}_{\mathfrak{s}}(y,\Omega) \approx \mathrm{diam}_{\mathfrak{s}} Q^n$ for $n \in N_y(\Omega)$. In the general case choose first $x' \in \overline{\Omega}$ such that $\mathrm{dist}_{\mathfrak{s}}(y,\Omega) = \|y-x'\|_{\mathfrak{s}}$ and split $\mathcal{E}^\alpha_y F - \Gamma^\alpha_{yx} F_x = \mathcal{E}^\alpha_y F - \Gamma^\alpha_{yx'} F_{x'} + \Gamma^\alpha_{yx}(\Gamma_{xx'} F_{x'} - F_x)$ and the statement follows due to $\|y-x'\|_{\mathfrak{s}} = \mathrm{dist}_{\mathfrak{s}}(y,\Omega) \leqslant \|y-x\|_{\mathfrak{s}}, \|x-x'\|_{\mathfrak{s}} \leqslant \|x-y\|_{\mathfrak{s}} + \|y-x'\|_{\mathfrak{s}} \lesssim \|y-x\|_{\mathfrak{s}}$. For the case 2 observe, similar as above, that it is sufficient to consider $x \in \overline{\Omega}^c$ with $\mathrm{dist}_{\mathfrak{s}}(x,\Omega) \leqslant 1$, since otherwise we can use the boundedness of the components $\mathcal{E}^\alpha F$, already shown above. Choose $y' \in \overline{\Omega}$ such that $\mathrm{dist}_{\mathfrak{s}}(x,\Omega) = \|x-y'\|_{\mathfrak{s}}$ and split

$$\begin{aligned}\mathcal{E}^\alpha_y F - \Gamma^\alpha_{yx}\mathcal{E}_x F &= F^\alpha_y - \Gamma^\alpha_{yy'} F_{y'} + \Gamma^\alpha_{yx}\Gamma_{xy'} F_{y'} - \Gamma^\alpha_{yx}\mathcal{E}_x F \\ &= F^\alpha_y - \Gamma^\alpha_{yy'} F_{y'} + \Gamma^\alpha_{yx} \sum_{n:\, x\in Q^n} \sum_{\nu_1+\nu_2<\gamma} \Gamma^{\nu_1}_{xp^n}(\Gamma_{p^n y'} F_{y'} - F_{p^n}) \star (\Phi^n)^{\nu_2}_x\,, \end{aligned} \tag{5.74}$$

where we used $\sum_{n:\, x\in Q^n} \Phi^n_x = \mathbf{1}$ in the second step. The desired estimate readily follows. The case 3 is then a consequence of 1 and 2 if we choose $\zeta \in \overline{\Omega}$ such that $\mathrm{dist}_{\mathfrak{s}}(\zeta,[x,y]) = \mathrm{dist}_{\mathfrak{s}}(\Omega,[x,y])$ and reshape $\mathcal{E}^\alpha_y F - \Gamma_{yx}\mathcal{E}_x F_x = \mathcal{E}^\alpha_y F - \Gamma_{y\zeta} F_\zeta + \Gamma_{y\zeta}(F_\zeta - \Gamma_{\zeta x}\mathcal{E}_x F)$. Let's now turn to case 4. Note that we now have

$$\mathrm{dist}_{\mathfrak{s}}([x,y],\Omega) \approx \mathrm{dist}_{\mathfrak{s}}(x,\Omega) \approx \mathrm{dist}_{\mathfrak{s}}(y,\Omega) \approx \mathrm{dist}_{\mathfrak{s}}(Q^n,\Omega)$$

for $n \in N_x(\Omega) \cup N_y(\Omega)$. First consider pairs x, y with $\mathrm{dist}_{\mathfrak{s}}(y, \Omega) \leqslant 1$ and $\mathrm{dist}_{\mathfrak{s}}(x, \Omega) \leqslant 1$. Choose $\zeta \in \overline{\Omega}$ such that $\mathrm{dist}_{\mathfrak{s}}(\zeta, [x, y]) = \mathrm{dist}_s(\Omega, [x, y])$ and reshape

$$\mathcal{E}^\alpha_y F - \Gamma^\alpha_{yx} \mathcal{E}_x F = \sum_{n \in N_x(\Omega) \cup N_y(\Omega)} \sum_{\mu_1 + \mu_2 = \alpha} \sum_{\mu_1 \leqslant \nu < \gamma} \Gamma^{\mu_1}_{yx} \Gamma^\nu_{xp^n} F_{p^n} \star ((\Phi^n_y)^{\mu_2} - \Gamma^{\mu_2}_{yx} (\Phi^n_x)^{<\gamma-\nu}) \tag{5.75}$$

$$\overset{(*)}{=} \sum_{\mu_1 + \mu_2 = \alpha} \sum_{\mu_1 \leqslant \nu < \gamma} \sum_{n \in N_x(\Omega) \cup N_y(\Omega)} \Gamma^{\mu_1}_{yx} (\Gamma^\nu_{xp^n} F_{p^n} - \Gamma^\nu_{x\zeta} F_\zeta) \star ((\Phi^n_y)^{\mu_2} - \Gamma^{\mu_2}_{yx} (\Phi^n_x)^{<\gamma-\nu}) \tag{5.76}$$

where we used γ-regularity of $(\mathcal{V}, \overline{\mathcal{T}})$ in the first equality and in $(*)$ the identity

$$\sum_{n \in N_x(\Omega) \cup N_y(\Omega)} \Phi^n(x) - \sum_{n \in N_x(\Omega) \cup N_y(\Omega)} \Gamma^{\mu_2}_{yx} \Phi^n(y) = \mathbf{1} - \mathbf{1} = 0$$

to sneak in a term $\Gamma^{\mu_1}_{yx} \Gamma^\nu_{x\zeta} F_\zeta$, independent of n. Since $\Gamma^{\mu_1}_{yx} (\Gamma^\nu_{xp^n} F_{p^n} - \Gamma^\nu_{x\zeta} F_\zeta) = \Gamma^{\mu_1}_{yx} (\Gamma^\nu_{xp^n} F_{p^n} - F^\nu_x + F^\nu_x - \Gamma^\nu_{x\zeta} F_\zeta)$ can be estimated by $\|y - x\|^{\nu - \mu_1} \|x - p^n\|^{\gamma - \nu} \lesssim \|y - x\|^{\nu - \mu_1} \mathrm{dist}_{\mathfrak{s}}(x, \Omega)^{\gamma - \nu} \approx \|y - x\|^{\nu - \mu_1} \mathrm{dist}_{\mathfrak{s}}(\Omega, [x, y])^{\gamma - \nu}$ we obtain together with $\|\Phi^n\|_{\mathcal{D}^{\gamma - \nu}(\mathbb{R}^d)} \lesssim \mathrm{dist}_{\mathfrak{s}}(Q^n, \Omega)^{\nu - \gamma} \approx \mathrm{dist}_{\mathfrak{s}}(\Omega, [x, y])^{\nu - \gamma}$ (for $n \in N_x(\Omega) \cup N_y(\Omega)$) the total bound

$$\sum_{\mu_1 + \mu_2 = \alpha} \sum_{\mu_1 \leqslant \nu < \gamma} \|y - x\|^{\nu - \mu_1}_{\mathfrak{s}} \mathrm{dist}_{\mathfrak{s}}(\Omega, [x, y])^{\gamma - \nu} \mathrm{dist}_{\mathfrak{s}}(\Omega, [x, y])^{\nu - \gamma} \|y - x\|^{\gamma - \nu - \mu_2}_{\mathfrak{s}}$$
$$\lesssim \|y - x\|^{\gamma - \alpha}_{\mathfrak{s}}.$$

It remains to analyze case 4 when either x or y has distance more than 1 from Ω, so that we now have $\mathrm{dist}_{\mathfrak{s}}([x, y], \Omega) > \frac{1}{2}$. We can assume that $\|x - y\|_{\mathfrak{s}} \leqslant 1$ since otherwise we can simply use the boundedness of the components of $\mathcal{E}F$, but then we have $\|x - p^n\|_{\mathfrak{s}} \lesssim 1$ for all terms in (5.75) that do not vanish. Estimating then $\|\Gamma^\nu_{xp^n} F_{p^n}\|_{\mathcal{T}_\nu} \lesssim 1$ yields for (5.75) the bound $\sum_{\mu_1 + \mu_2 = \alpha} \sum_{\mu_1 \leqslant \nu < \gamma} \|y - x\|^{\nu - \mu_1}_{\mathfrak{s}} \|y - x\|^{\gamma - \nu - \mu_2}_{\mathfrak{s}} \lesssim \|y - x\|^{\gamma - \alpha}_{\mathfrak{s}}$, due to $\|\Phi^n\|_{\mathcal{D}^{\gamma - \nu}(\mathbb{R}^d)} \lesssim 1$ for $n \in N_x(\Omega) \cup N_y(\Omega)$.

To prove (5.71) for $\mathcal{E}F = \mathcal{E}^\Gamma_\Omega F$, $\hat{\mathcal{E}}F := \mathcal{E}^{\hat{\Gamma}}_\Omega F$ we have to estimate for $\alpha \in A$ the objects

$$\|\mathcal{E}^\alpha_x F - \hat{\mathcal{E}}^\alpha_x F\|_{\mathcal{T}_\alpha}, \; \|\mathcal{E}^\alpha_y F - \Gamma^\alpha_{yx} \mathcal{E}_x F - (\hat{\mathcal{E}}^\alpha_y F - \hat{\Gamma}^\alpha_{yx} \hat{\mathcal{E}}_x F)\|_{\mathcal{T}_\alpha}$$

The first norm can once more be bounded similar as in (5.72), for the second quantity we use once more (5.73), (5.74) and (5.75)/(5.76) depending on the position of x, y. □

Chapter 6

Schauder theory for singular SPDEs based on paraproducts

In this chapter we give an application of the theory presented in Chapter 5. We want to prove Schauder estimates for singular SPDEs of the form

$$a(D)u := (\partial_t - p(D'))u = F(u, \xi),\ u(0) = u_0 \tag{6.1}$$

on $[0, T] \times \mathbb{R}^{d-1}$. Here, F is some nonlinearity that might also act non-local on u by depending on derivatives $\partial_x u$ for instance. The object $p(D')$ denotes some homogeneous polynomial of spatial derivatives of even degree θ. Let us denote the Green's function of the operator $a(D)$ by $\mathscr{A}$.

We want to describe the solutions of (6.1) via a regularity structure as in Chapter 2. In [Hai14] similar equations such as (6.1) were considered. We here give a distinct proof of the Schauder estimates for modelled distributions using the machinery we introduced in Chapter 5.

However, we will skip one important step in the solution theory for (6.1) via regularity structures: We will *not* give a renormalization theory for the considered models as in [Hai14, Section 9] or [BHZ16, CH16]. This is merely out of convenience, as such a step could probably be done in some similar way as in the references but would be quite elaborate without shedding any light on the usefulness of the theory from Chapter 5.

We will therefore instead assume we are already given a model (Π, Γ) on some regularity structure $\mathscr{T} = (A, \mathcal{T}, G)$ and show how to build a map

$$\mathcal{K} : \mathcal{D}^{[\eta,\gamma]}(\Omega^T; \mathcal{T}, \Gamma) \to \mathcal{D}^{[\eta+\theta,\gamma+\theta]}(\Omega^T; \mathcal{T}, \Gamma) \tag{6.2}$$

that corresponds to integration against the Green's function $\mathscr{A}$ of $a(D)$ on the level of modelled distributions. The map (6.2) is constructed by the main result of this chapter, Theorem 6.2.3 below. Actually, in order to obtain estimates that can be “closed” to prove existence of solutions one has to subtract an (arbitrarily) small $\kappa > 0$ from the parameters of the codomain of $\mathcal{K}$. As in Chapter 4 the key role in this chapter will be played by a commutation result for paraproducts with the differential operator, that is $a(D)$ in our case. This will be the content of Theorem 6.1.9 below. We further show in Section 6.1 how to integrate a model against the Green’s function $\mathscr{A}$, where we exploit the properties of the Fourier transform of $\mathscr{A}$, which is in contrast to [Hai14]. We also give a version of the “extension theorem” [Hai14, Theorem 5.14] for integration against $\mathscr{A}$ in Theorem 6.1.8.

As in Chapter 5 we will solely work with classical tempered distributions, instead of ultra-distributions, as we do not consider problems involving weights.

This chapter will be content of [MP18].

Notation

Most of the notation of this chapter will be taken from Chapter 5. However, we will often need to distinguish between time and space. Since $\mathbb{R}^d$ denotes, as in Chapter 5, space-time it will be convenient to split

$$x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1} ,$$

where x_1 should be read as time, while x' represents the location in space. We will use a similar notation for Fourier multipliers:

Recall that we call $\sigma(D)$ a multiplier operator for $\sigma \in C^\infty(\mathbb{R}^d;\mathbb{C})$, say with polynomial growing derivatives, if for any Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}^d;\mathbb{C})$

$$\sigma(D)\varphi = \mathcal{F}^{-1}_{\mathbb{R}^d}(\sigma \cdot \mathcal{F}_{\mathbb{R}^d}\varphi) .$$

By duality this gives again a Schwartz function, so that $\sigma(D)$ acts on Schwartz distributions by duality. σ is known as the symbol for $\sigma(D)$. We can also define for $\sigma \in C^\infty(\mathbb{R}^{d-1};\mathbb{C})$, again with polynomial growing derivatives, $\varphi \in \mathcal{S}(\mathbb{R}^d;\mathbb{C})$ and $x = (x_1, x') \in \mathbb{R}^d$

$$(\sigma(D')\varphi)(x) = \mathcal{F}^{-1}_{\mathbb{R}^{d-1}}(\sigma \cdot \mathcal{F}_{\mathbb{R}^{d-1}}\varphi(x_1, \cdot))(x')$$

As this gives once more an element of $\mathcal{S}(\mathbb{R}^d)$, we can also extend $\sigma(D')$ to $\mathcal{S}'(\mathbb{R}^d)$ by duality.

6.1 Fourier multipliers

In this section we show how (homogeneous) Fourier multipliers $a(D)$ can be brought into the framework presented in Sections 2.3, 5.1 and 5.2. In particular Subsection

6.1.1 of this Chapter will appeal probably, to a reader familiar with the basic concepts of regularity structures, as a natural translation of the concept of integration against singular kernels to the Fourier world. Instead of decomposing the kernel in compactly supported functions we decompose its spectrum instead, which turns out to be a useful technique when we turn to the interplay with paraproducts in Subsection 6.1.2, which is in turn the key estimate for our Schauder theory in Section 6.2. We will fix throughout this section some scaling vector

$$\mathfrak{s} \in [1,\infty)^d .$$

Definition 6.1.1. *Given $\theta \in \mathbb{R}$ we say that an operator $a(D)$ with symbol $a \in C^\infty(\mathbb{R}^d\backslash\{0\};\mathbb{C})$, satisfying $|a(z)| > 0$ for $z \in \mathbb{R}^d\backslash\{0\}$, is a* Fourier multiplier of order θ *and write $a(D) \in \mathcal{M}^\theta_{\mathfrak{s}}(\mathbb{R}^d)$ if for any $x \in \mathbb{R}^d$ and $\lambda > 0$*

$$a(\lambda^{\mathfrak{s}} x) = \lambda^\theta \cdot a(x) .$$

We set for $j \geqslant 0$

$$\mathscr{A}^j = \mathcal{F}^{-1}_{\mathbb{R}^d}\left(\frac{\varphi_j}{a}\right) , \tag{6.3}$$

where $(\varphi_j)_{j\geqslant -1}$ is some dyadic, anisotropic partition of unity as in Section 2.1 , constructed with the same scaling vector $\mathfrak{s}$. We call

$$\mathscr{A} := \sum_{j\geqslant 0} \mathscr{A}^j . \tag{6.4}$$

the kernel associated to $a(D)$.

Remark 6.1.2. *Note that $\mathscr{A}$ might be only existent in $\mathcal{S}'(\mathbb{R}^d)$ and we will therefore use $\mathscr{A}$ solely in formal expressions which are accompanied by a rigorous definition. $\mathscr{A}$ should be thought of as some replacement for K as used in [Hai14, Section 5.1], whereas the $\mathscr{A}_j$ replace the decomposition $K = \sum_n K_n$ used there.*

The restriction to $j \geqslant 0$ is no typo! It is not clear that (6.3) would be well-defined in general for $j = -1$. Moreover, we ensure by the exclusion of the "0 modes" that for $k \in \mathbb{N}^d$

$$\int_{\mathbb{R}^d} \mathrm{d}x\, \mathscr{A}(x)\, x^k := \sum_{j\geqslant 0} \int_{\mathbb{R}^d} \mathrm{d}u\, \mathscr{A}^j(x)\, x^k = 0 ,$$

which is in analogy to [Hai14, Assumption 5.4].

In a certain sense $\mathscr{A}$ can be regarded as the "inverse" of $a(D)$.

A typical example one could have in mind for $a(D) \in \mathcal{M}^{\theta}_{\mathfrak{s}}(\mathbb{R}^d)$ would be the fractional Laplacian $a(D) = (-\Delta)^{\frac{\theta}{2}}$ for $\theta > 0$ with $a(\xi) = |\xi|^{\theta}$ (and isotropic scaling $\mathfrak{s} = (1, \ldots, 1)$) or for $\theta = 2$ the heat operator

$$a(D) = \partial_t - \Delta \,,$$

where now $a(\xi) = 2\pi\imath\xi_1 + |\xi'|^2$ and $\mathfrak{s} = (2, 1, \ldots, 1)$.

We use the description "of order θ" for $a(D)$ because of the properties that are described by the following lemma.

Lemma 6.1.3. *For $\alpha \in \mathbb{R}$, $\theta \in \mathbb{R}$ and $a \in \mathcal{M}^{\theta}_{\mathfrak{s}}(\mathbb{R}^d)$ we have*

$$\|\mathscr{A} * f\|_{\mathcal{C}^{\alpha+\theta}_{\mathfrak{s}}(\mathbb{R}^d)} \lesssim \|f\|_{\mathcal{C}^{\alpha}_{\mathfrak{s}}(\mathbb{R}^d)} \,, \tag{6.5}$$

and further, provided that $a \in C^{\infty}(\mathbb{R}^d)$ or $\operatorname{supp} \mathcal{F}_{\mathbb{R}^d} f \subseteq B(0, \delta)^c$ for some $\delta > 0$,

$$\|a(D) f\|_{\mathcal{C}^{\alpha-\theta}_{\mathfrak{s}}(\mathbb{R}^d)} \lesssim \|f\|_{\mathcal{C}^{\alpha}_{\mathfrak{s}}(\mathbb{R}^d)} \,.$$

The first estimate (6.5) should be read in the sense that the sum $\sum_{j \geqslant 0} \mathscr{A}_j * f$ is convergent in $\mathcal{S}'(\mathbb{R}^d)$ and its limit is contained in $\mathcal{C}^{\alpha+\theta}_{\mathfrak{s}}(\mathbb{R}^d)$ with the claimed bound.

Proof. Consider the Littlewood-Paley-blocks for the sequence $\sum_{j=0}^{N} \mathscr{A}^j * f$

$$\Delta_i \left(\sum_{j=0}^{N} \mathscr{A}^j * f \right) = \sum_{0 \leqslant j \leqslant N :\, j \sim i} \mathscr{A}^j * \Delta_i f \,,$$

where we used spectral support properties together with Lemma 2.1.12. Using Lemma 6.1.4 below we see via Young's inequality that for $i \sim j$

$$\|\mathscr{A}^j * \Delta_i f\|_{L^{\infty}(\mathbb{R}^d)} \lesssim \|\mathscr{A}^j\|_{L^1(\mathbb{R}^d)} \|\Delta_i f\|_{L^{\infty}(\mathbb{R}^d)} \lesssim 2^{-j(\alpha+\theta)} \|f\|_{\mathcal{C}^{\alpha}_{\mathfrak{s}}(\mathbb{R}^d)}$$

The sequence $\sum_{j=0}^{N} \mathscr{A}^j * f$ is therefore uniformly bounded in $\mathcal{C}^{\alpha}_{\mathfrak{s}}(\mathbb{R}^d)$ and hence a Cauchy sequence in $\mathcal{C}^{\alpha'}_{\mathfrak{s}}(\mathbb{R}^d)$, $\alpha' < \alpha$ and convergent (for completeness of anisotropic Besov spaces see [Tri06, Theorem 5.3]). In particular it is convergent in $\mathcal{S}'(\mathbb{R}^d)$ and, by the estimates above, bounded in $\mathcal{C}^{\alpha+\theta}_{\mathfrak{s}}(\mathbb{R}^d)$ by $\|f\|_{\mathcal{C}^{\alpha}_{\mathfrak{s}}(\mathbb{R}^d)}$.

For the statement concerning $a(D)$ compare [BCD11, Proposition 2.30] (which is for isotropic, homogeneous Besov spaces, but easily translated to the case considered here). $\square$

6.1.1 Integration of the model

The central role in this subsection will be played by the Schwartz functions $\mathscr{A}^j$ we introduced in Definition 6.1.1. The behaviour of $\mathscr{A}^j$ is described by the following lemma.

Lemma 6.1.4. *The functions $(\mathscr{A}^j)_{j\geqslant 0}$ defined by (6.3) are contained in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and satisfy the following scaling property for $j \geqslant 0$:*

$$\mathscr{A}^j = 2^{j(|\mathfrak{s}|-\theta)}\mathscr{A}^0(2^{j\mathfrak{s}}\cdot)\,.$$

Proof. Every $\mathscr{A}^j$ is a Schwartz function as it is the Fourier transform of the smooth, compactly supported function φ_j/a. The scaling property follows by substitution and the homogeneity of a

$$\mathscr{A}^j(x) = \int \mathrm{d}\xi\, e^{2\pi\imath x\cdot\xi}\frac{\varphi(2^{-j\mathfrak{s}}\xi)}{a(\xi)} = \int \mathrm{d}\xi\, 2^{j|\mathfrak{s}|}\, e^{2\pi\imath 2^{j\mathfrak{s}}x\cdot\xi}\frac{\varphi(\xi)}{2^{j\theta}a(\xi)} = 2^{j(|\mathfrak{s}|-\theta)}\,\mathscr{A}^0(2^{j\mathfrak{s}}x)\,.$$

□

As a consequence we get the following lemma.

Lemma 6.1.5. *Let $(\mathscr{A}^j)_{j\geqslant 0}$ be as in (6.3). Given a model (Π,Γ) on a regularity structure $\mathscr{T} = (A,\mathcal{T},G)$ we have for $x\in\mathbb{R}^d$, $k\in\mathbb{N}^d$, $\alpha\in A$, $\tau\in\mathcal{T}_\alpha$ and $j\geqslant 0$*

$$|\Pi_x\tau(\partial^k\mathscr{A}^j_{x-\cdot})| \lesssim K\cdot 2^{-j(\alpha+\theta-|k|_\mathfrak{s})}\,, \tag{6.6}$$

where K is a polynomial in $|\!|\!|(\Pi,\Gamma)|\!|\!|_\gamma$ and where $\gamma\in\mathbb{R}$ can be chosen arbitrary with the only restriction that $\alpha<\gamma$. Given a $\varphi\in\mathcal{S}(\mathbb{R}^d)$ we also have for $\lambda\in(0,1]$, $2^{-j}\lesssim\lambda$

$$\Big|\int \mathrm{d}u\,\varphi^\lambda_{u-x}\,\Pi_x\tau(\mathscr{A}^j_{u-\cdot})\Big| \lesssim K\cdot[\varphi]\cdot 2^{-j\theta}\,\lambda^\alpha \tag{6.7}$$

with $[\varphi]$ as in Lemma 2.3.10. Given a second model $(\hat\Pi,\hat\Gamma)$ we further have the estimates

$$\big|\big((\Pi_x-\hat\Pi_x)\tau\big)(\partial^k\mathscr{A}^j_{x-\cdot})\big| \lesssim K\cdot |\!|\!|(\Pi,\Gamma);(\hat\Pi,\hat\Gamma)|\!|\!|_\gamma\cdot 2^{-j(\alpha+\theta-|k|_\mathfrak{s})}\,,$$
$$\Big|\int\mathrm{d}u\,\varphi^\lambda_{u-x}\,\big((\Pi_x-\hat\Pi_x)\tau\big)(\mathscr{A}^j_{u-\cdot})\Big| \lesssim K\cdot[\varphi]\cdot|\!|\!|(\Pi,\Gamma);(\hat\Pi,\hat\Gamma)|\!|\!|_\gamma\cdot 2^{-j(\alpha+\theta-|k|_\mathfrak{s})}\,.$$

Proof. By Lemma 6.1.4 we have

$$\Pi_x\tau(\partial^k\mathscr{A}^j_{x-\cdot}) = 2^{j(|k|_\mathfrak{s}-\theta)}\,\Pi_x\tau\big(\partial^k\mathscr{A}^0(2^{j\mathfrak{s}}(x-\cdot))\big)$$

so that (6.6) follows by Lemma 2.3.10. To see (6.7), we can reshape (to make this step rigorous approximate $\Pi_x\tau$ by a smooth sequence)

$$\int \mathrm{d}u\, \varphi^\lambda_{u-x}\, \Pi_x\tau(\mathscr{A}^j_{u-\cdot}) = \Pi_x\tau\left(\int \mathrm{d}u\, \mathscr{A}^j_{u-\cdot}\, \varphi^\lambda_{u-x}\right) = \int \mathrm{d}u\, \mathscr{A}^j_u\, \Pi_x\tau\big(\varphi^\lambda_{\cdot+u-x}\big)\,.$$

Using now $\Pi_x\tau = \Pi_{x-u}\Gamma_{x-u,x}\tau = \sum_{\alpha'\in A\colon\alpha'\leqslant\alpha}\Pi_{x-u}\Gamma^{\alpha'}_{x-u,x}\tau$, Lemma 6.1.4 and Lemma 2.3.10 we can bound the right hand side by

$$\Big|\sum_{\alpha'\in A\colon\alpha'\leqslant\alpha}\int \mathrm{d}u\, \mathscr{A}^j_u\, \Pi_{x-u}\Gamma^{\alpha'}_{x-u,x}\tau\big(\varphi^\lambda_{\cdot-(x-u)}\big)\Big| \lesssim \sum_{\alpha'\in A\colon\alpha'\leqslant\alpha} K\cdot[\varphi]\, 2^{-j(\theta+\alpha-\alpha')}\lambda^{\alpha'}\,,$$

so that we conclude using $2^{-j(\alpha-\alpha')} \lesssim \lambda^{\alpha-\alpha'}$. The distance estimates follow by essentially the same arguments, after an easy modification of Lemma 2.3.10. $\square$

If our regularity structure $\mathscr{T} = (A, \mathcal{T}, G)$ satisfies Assumption 2.3.12 we define, similarly to [Hai14], a linear operator $\mathcal{J} : \mathbb{R}^d \times \mathcal{T} \to \overline{\mathcal{T}} \subseteq \mathcal{T}$ given for $\tau \in \mathcal{T}_\alpha$, $\alpha\in A$ by

$$\begin{aligned}\mathcal{J}(x)\tau &:= \sum_{k\in\mathbb{N}^d_{<\alpha+\theta}} \frac{X^k}{k!}\int \Pi_x\tau(\mathrm{d}z)\,\partial^k\mathscr{A}_{x-z}\\ &:= \sum_{j\geqslant 0}\mathcal{J}_j(x)\tau := \sum_{k\in\mathbb{N}^d_{<\alpha+\theta}}\frac{X^k}{k!}\sum_{j\geqslant 0}\Pi_x\tau\left(\partial^k\mathscr{A}^j_{x-\cdot}\right)\end{aligned} \tag{6.8}$$

Using Lemma 6.1.5 one easily sees that the sum over $j \geqslant 0$ is convergent for every $x\in\mathbb{R}^d$ so that $\mathcal{J}(x)$ is well-defined. The component of the operator $\mathcal{J}(x)$ in $\mathcal{T}_\beta = \overline{\mathcal{T}}_\beta$ for $\beta\in\overline{A} = |\mathbb{N}^d|_{\mathfrak{s}}$ will as usual be denoted by an upper index:

$$\mathcal{J}^\beta(x)\tau := (\mathcal{J}(x)\tau)^\beta = \sum_{k\in\mathbb{N}^d\colon|k|=\beta} X^k\cdot(\mathcal{J}(x)\tau)^{X^k} =: \sum_{k\in\mathbb{N}^d\colon|k|=\beta} X^k\cdot\mathcal{J}^{X^k}(x)\tau\,.$$

We use the same notation for the operators $(\mathcal{J}_j)_{j\geqslant 0}$.

Next, we will introduce, as in [Hai14], an abstract integration map $\mathcal{I}$.

Definition 6.1.6. *Let $\mathscr{T} = (A, \mathcal{T}, G)$ be a regularity structure satisfying Assumption 2.3.12. An abstract integration map of order θ on a sector $\mathcal{V}\subseteq\mathcal{T}$ is a linear map $\mathcal{I} : \mathcal{V}\to\mathcal{T}\backslash\overline{\mathcal{T}}$ with the properties:*

- *For any $\tau\in\mathcal{V}_\alpha$, $\alpha\in A$ one has $\mathcal{I}\tau\in\mathcal{T}_{\alpha+\theta}$ (with $\mathcal{I}\tau = 0$ if $\alpha+\theta\notin A$).*
- *$\mathcal{I}(\mathcal{V}\cap\overline{\mathcal{T}})\subseteq\{0\}$.*
- *For $\Gamma\in G$, $\tau\in\mathcal{V}$ $\mathcal{I}\Gamma\tau - \Gamma\mathcal{I}\tau\in\overline{\mathcal{T}}$.*

We denote by $\mathcal{I}^\alpha \tau := (\mathcal{I}\tau)^\alpha$ *the component of* $\mathcal{I}\tau$ *within* $\mathcal{T}_\alpha$ *for* $\alpha \in A$ *and* $\tau \in \mathcal{T}$ *and write* $\mathcal{I}^{<\beta} := \sum_{\alpha\in A:\,\alpha<\beta} \mathcal{I}^\alpha$ *for* $\beta \in \mathbb{R}$. *Let* $a(D) \in \mathcal{M}_s^\theta(\mathbb{R}^d)$ *be a Fourier multiplier of order* θ *as in Definition 6.1.1, let* $\mathscr{A}$ *be the associated kernel and define* $\mathcal{J}$ *as in* (6.8). *We then say that a model* (Π, Γ) *realizes* $\mathscr{A}$ *for* $\mathcal{I}$ *on* $\mathcal{V}$ *if for any* $\tau \in \mathcal{V}$, $x \in \mathbb{R}^d$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$

$$\Pi_x \mathcal{I}\tau(\varphi) = \int \mathrm{d}u\, \varphi(u) \cdot \Pi_x \tau(\mathscr{A}_{u-\cdot}) - \Pi_x \left(\mathcal{J}(x)\tau\right)(\varphi)\,. \tag{6.9}$$

Remark 6.1.7. *The identity* (6.9) *should really be read for* $\tau \in \mathcal{T}_\alpha$ *with* $\alpha \in A_{\mathcal{V}}$ *as*

$$\Pi_x \mathcal{I}\tau(\varphi) = \sum_{j\geqslant 0} \int \mathrm{d}u\, \varphi(u) \cdot \Pi_x \tau\left(R^{\alpha+\theta}_{x-\cdot;u-x} \mathscr{A}^j\right) \tag{6.10}$$

with $R^{\alpha+\theta}_{x-\cdot;u-x}$ *as in* (2.22). *The fact that this sum is convergent is a byproduct of the proof of Theorem 6.1.8 below.*

We can now formulate an analogous result to the "extension theorem" in [Hai14, Theorem 5.14] for Fourier multipliers.

Theorem 6.1.8. *Let* $a(D) \in \mathcal{M}_s^\theta(\mathbb{R}^d)$ *be a Fourier multiplier of order* θ *and let* $\mathscr{A}$ *be the associated kernel. Assume* $\mathscr{T} = (A, \mathcal{T}, G)$ *is a regularity structure, satisfying Assumption 2.3.12, with a model* (Π, Γ) *and with two (possibly empty) sectors* $\mathcal{W} \subseteq \mathcal{V}$, *where* $\mathcal{W}$ *is a subsector of* $\mathcal{V}$ *and where* $\mathcal{V}$ *has the property that for* $\alpha \in A_{\mathcal{V}} \backslash |\mathbb{N}^d|_{\mathfrak{s}}$ *once has* $\alpha + \theta \notin |\mathbb{N}^d|_{\mathfrak{s}}$. *Assume further that there is an abstract integration map* $\mathcal{I}$ *of order* θ *on* $\mathcal{W}$ *given, as well as a model* (Π, Γ) *that realizes* $\mathscr{A}$ *for* $\mathcal{I}$ *on* $\mathcal{W}$.

Then, there is a regularity structure $\tilde{\mathscr{T}} = (\tilde{A}, \tilde{\mathcal{T}}, \tilde{G}) \supseteq \mathscr{T}$ *with a model* $(\tilde{\Pi}, \tilde{\Gamma})$ *which extends* $\mathscr{T}$ *and* (Π, Γ) *together with an abstract integration map* $\tilde{\mathcal{I}}$ *that extends* $\mathcal{I}$ *from* $\mathcal{W}$ *to* $\mathcal{V}$ *such that* $\tilde{\Pi}$ *realizes* $\mathscr{A}$ *for* $\tilde{\mathcal{I}}$ *on* $\mathcal{V}$.

Moreover we have for $x, y \in \mathbb{R}^d$

$$\tilde{\Gamma}_{yx}(\tilde{\mathcal{I}} + \tilde{\mathcal{J}}(x)) = (\tilde{\mathcal{I}} + \tilde{\mathcal{J}}(y))\tilde{\Gamma}_{yx} \tag{6.11}$$

and the bound $\|(\tilde{\Pi}, \tilde{\Gamma})\|_{\gamma+\theta} \lesssim K$ *for* $\gamma \in \mathbb{R}$, *where* K *is a polynomial in* $\|(\Pi, \Gamma)\|_\gamma$. *Given a second model* $(\hat{\Pi}, \hat{\Gamma})$ *with corresponding extension* $(\tilde{\hat{\Pi}}, \tilde{\hat{\Gamma}})$ *we further have the estimate*

$$\|(\tilde{\Pi}, \tilde{\Gamma}); (\tilde{\hat{\Pi}}, \tilde{\hat{\Gamma}})\|_{\gamma+\theta} \lesssim K \cdot \|(\Pi, \Gamma); (\hat{\Pi}, \hat{\Gamma})\|_\gamma\,, \tag{6.12}$$

where K *is once more a polynomial in* $\|(\Pi, \Gamma)\|_\gamma$.

Proof. The construction of $\tilde{\mathscr{T}}$ and of $\mathcal{I}$ on $\mathcal{V}$ follows along the lines of the proof of Theorem 5.14 in [Hai14], where for any $\alpha \in A_{\mathcal{V}} \backslash |\mathbb{N}^d|_{\mathfrak{s}}$ a copy of some complement

$\mathcal{V}_\alpha \backslash \mathcal{W}_\alpha$ of $\mathcal{W}_\alpha$ within $\mathcal{V}_\alpha$ (as in Definition 2.3.4) is added to the regularity structure and identified with the image of $\mathcal{V}_\alpha \backslash \mathcal{W}_\alpha$ under $\mathcal{I}$ (and associated the homogeneity $\alpha + \theta$). We will identify $\mathcal{I}$ with its extension $\tilde{\mathcal{I}}$ and simply write $\mathcal{I}$.

The only aspect which remains to be checked is that $\tilde{\Pi}_x \mathcal{I}\tau$ and $\tilde{\Gamma}_{yx}\mathcal{I}\tau$ fixed by (6.9) and (6.11) for $\tau \in \mathcal{V}_\alpha$ with $\alpha \in A_{\mathcal{V}} \backslash |\mathbb{N}^d|_{\mathfrak{s}}$ to be

$$\tilde{\Gamma}_{yx}\mathcal{I}\tau = \mathcal{I}\Gamma_{yx}\tau + (\mathcal{J}(y)\Gamma_{yx}\tau - \Gamma_{yx}\mathcal{J}(x)\tau)\,, \tag{6.13}$$

$$\tilde{\Pi}_x \mathcal{I}\tau(\varphi) = \sum_{j \geqslant 0} \int \mathrm{d}u\, \varphi(u) \cdot \Pi_x \tau \left(R^{\alpha+\theta}_{x-\cdot;u-x} \mathscr{A}^j\right)\,, \tag{6.14}$$

satisfy the relations and bounds from Definition 2.3.9. Note that we used that $\mathcal{J}$ and $\tilde{\mathcal{J}}$ coincide on $\mathcal{V} \subseteq \mathcal{T}$ and thus in (6.13).

The proof of the relations $\tilde{\Pi}_x = \tilde{\Pi}_y \tilde{\Gamma}_{yx}$, $\tilde{\Gamma}_{yx}\tilde{\Gamma}_{xz} = \tilde{\Gamma}_{yz}$ is again just the same as in [Hai14] so that we are left with the bounds (2.38) and (2.39). We will write, as in the claim, K for a polynomial in $|\!|\!|(\Pi, \Gamma)|\!|\!|_\gamma$, that might change from line to line.

Consider $\alpha \in A_{\mathcal{V}} \backslash |\mathbb{N}^d|_{\mathfrak{s}}$, $\tau \in \mathcal{V}_\alpha$ and $\gamma \in \mathbb{R}$ such that $\alpha < \gamma$. We show (2.38) first, so that we have to bound the components of $\Gamma_{yx}\mathcal{I}\tau$ for y, $x \in \mathbb{R}^d$.

We will bound the two terms in (6.13) separately.

The first term, that is $\mathcal{I}\Gamma_{yx}\tau$, can only have components in the spaces $\tilde{\mathcal{T}}_{\beta+\theta}$, where $\beta \in A_{\mathcal{V}} \backslash |\mathbb{N}^d|_{\mathfrak{s}}$ with $\beta \leqslant \alpha$. We can bound each of these components by

$$\begin{aligned} \|\mathcal{I}^{\beta+\theta}\Gamma_{yx}\tau\|_{\tilde{\mathcal{T}}_{\beta+\theta}} &\overset{(*_1)}{=} \|\mathcal{I}\Gamma^\beta_{yx}\tau\|_{\tilde{\mathcal{T}}_{\beta+\theta}} \overset{(*_2)}{\lesssim} \|\Gamma^\beta_{yx}\tau\|_{\tilde{\mathcal{T}}_\beta} \overset{(*_3)}{\lesssim} \|\Gamma^\beta_{yx}\tau\|_{\mathcal{T}_\beta} \\ &\leqslant K \cdot \|y - x\|_{\mathfrak{s}}^{\alpha-\beta} = K \cdot \|y - x\|_{\mathfrak{s}}^{\alpha+\theta-(\beta+\theta)}\,, \end{aligned}$$

where we used in $(*_1)$ and $(*_2)$ the properties of an abstract integration map from Definition 6.1.6. In $(*_3)$ we used the continuity of the embedding $\mathcal{T}_{\beta+\theta} \subseteq \tilde{\mathcal{T}}_{\beta+\theta}$. Note that this is the right bound since $\mathcal{I}\tau \in \tilde{\mathcal{T}}_{\alpha+\theta}$.

The second term in (6.13), that is $\mathcal{J}(y)\Gamma_{yx}\tau - \Gamma_{yx}\mathcal{J}(x)\tau$, contributes only on the polynomial subspace $\overline{\mathcal{T}}$. Consequently, we have to bound for $k \in \mathbb{N}^d$ with $|k|_{\mathfrak{s}} \leqslant \alpha + \theta$ (if any) and $j \geqslant 0$ objects like

$$\mathcal{J}^{X^k}_j(y)\Gamma_{yx}\tau - \Gamma^{X^k}_{yx}\mathcal{J}_j(x)\tau \tag{6.15}$$

Since we know by our assumption on $\mathcal{V}$ that $\alpha + |\theta| \notin |\mathbb{N}^d|_{\mathfrak{s}}$, we have in particular $|k|_{\mathfrak{s}} < \alpha + \theta$. We first rewrite (using $\Pi_x\tau = \sum_{\alpha' \leqslant \alpha} \Pi_y \Gamma^{\alpha'}_{yx}\tau$ in the second line)

$$\mathcal{J}^{X^k}_j(y)\Gamma_{yx}\tau = \sum_{\alpha' : \alpha' \leqslant \alpha} \mathcal{J}^{X^k}_j(y)\Gamma^{\alpha'}_{yx}\tau = \frac{1}{k!} \sum_{\alpha' : \alpha' \leqslant \alpha \,\&\, \alpha' + \theta > |k|_{\mathfrak{s}}} \int \Pi_y \Gamma^{\alpha'}_{yx}\tau(\mathrm{d}u)\, \partial^k \mathscr{A}^j_{y-u} \tag{6.16}$$

$$= \frac{1}{k!} \int \Pi_x\tau(\mathrm{d}u)\, \partial^k \mathscr{A}^j_{y-u} - \frac{1}{k!} \sum_{\alpha' : \alpha' \leqslant \alpha \,\&\, \alpha' + \theta < |k|_{\mathfrak{s}}} \int \Pi_y \Gamma^{\alpha'}_{yx}\tau(\mathrm{d}u) \partial^k \mathscr{A}^j_{y-u}\,. \tag{6.17}$$

where the sums run over $\alpha' \in A_{\mathcal{V}}$ and where we used in the second line that $\partial^k \mathscr{A}^j$ has spectral support away from 0 to erase a possible polynomial contribution coming from $\alpha' + \theta = |k|_{\mathfrak{s}}$ (which implies $\alpha' \in |\mathbb{N}^d|_{\mathfrak{s}}$ by our assumption on $\mathcal{V}$). Assume first $\|y-x\|_{\mathfrak{s}} \leqslant 1$ and pick $j' \geqslant 0$ such that $2^{-j'-1} < \|y-x\|_{\mathfrak{s}} \leqslant 2^{-j'}$.

We first consider $0 \leqslant j \leqslant j'$ and bound the sum of the terms coming from the second term in (6.17) via Lemma 6.1.5 by

$$\sum_{j \leqslant j'} \sum_{\alpha' :\, \alpha' \leqslant \alpha \,\&\, \alpha'+\theta < |k|_{\mathfrak{s}}} \|y-x\|_{\mathfrak{s}}^{\alpha-\alpha'} \cdot 2^{j(|k|_{\mathfrak{s}} - (\alpha'+\theta))} \lesssim \|y-x\|_{\mathfrak{s}}^{\alpha+\theta-|k|_{\mathfrak{s}}} .$$

We are in (6.15) then only left with terms of the form

$$\begin{aligned}
&\frac{1}{k!}\int \Pi_x\tau(\mathrm{d}u)\left[\partial^k \mathscr{A}^j_{y-u} - \sum_{l :\, |l|_{\mathfrak{s}} < \alpha+\theta-|k|_{\mathfrak{s}}} \frac{(y-x)^l}{l!}\partial^{k+l}\mathscr{A}^j_{x-u}\right] \\
=&\frac{1}{k!}\sum_{l \in \mathbb{N}^d_{>\alpha+\theta-|k|_{\mathfrak{s}}}} (y-x)^l \int_0^1 \mathrm{d}t\,(1-t)^{l_{\mathfrak{m}(l)}-1} \int \Pi_x\tau(\mathrm{d}u)\,\partial^{k+l}\mathscr{A}^j_{x+v_t^k(y-x)-u} ,
\end{aligned}$$

where we applied Lemma 2.1.20 in the second step. Writing
$\Pi_x\tau = \sum_{\alpha' :\, \alpha' \leqslant \alpha} \Pi_{x+v_t^k(y-x)}\,\Gamma^{\alpha'}_{x+v_t^k(y-x),x}\tau$ we then obtain via Lemma 6.1.5 the upper bound

$$K\frac{1}{k!}\sum_{l \in \mathbb{N}^d_{>\alpha+\theta-|k|_{\mathfrak{s}}}} \sum_{\alpha' :\, \alpha' \leqslant \alpha} \|y-x\|_{\mathfrak{s}}^{|l|_{\mathfrak{s}}+\alpha-\alpha'}\, 2^{-j(\alpha'+\theta-|k|_{\mathfrak{s}}-|l|_{\mathfrak{s}})}$$

which summed over $j \leqslant j'$ yields the bound

$$\begin{aligned}
&K\frac{1}{k!}\sum_{l \in \mathbb{N}^d_{>\alpha+\theta-|k|_{\mathfrak{s}}}} \sum_{\alpha' :\, \alpha' \leqslant \alpha} \|y-x\|_{\mathfrak{s}}^{|l|_{\mathfrak{s}}+\alpha-\alpha'} \sum_{j \leqslant j'} 2^{-j(\alpha'+\theta-|k|_{\mathfrak{s}}-|l|_{\mathfrak{s}})} \\
&\lesssim K\|y-x\|_{\mathfrak{s}}^{\alpha+\theta-|k|_{\mathfrak{s}}} .
\end{aligned}$$

We now show $\left|\sum_{j>j'} \mathcal{J}_j^{X^k}(y)\Gamma_{yx}\tau - \Gamma_{yx}^{X^k}\mathcal{J}_j(x)\tau\right| \lesssim K\,\|y-x\|_{\mathfrak{s}}^{\alpha+\theta-|k|_{\mathfrak{s}}}$. First, by applying Lemma 6.1.5 to (6.16), we have

$$\begin{aligned}
\Big|\sum_{j>j'} \mathcal{J}_j^{X^k}(y)\Gamma_{yx}\tau\Big| &\lesssim K\sum_{j>j'} \sum_{\alpha' :\, \alpha' \leqslant \alpha \,\&\, \alpha'+\theta > |k|_{\mathfrak{s}}} \|y-x\|_{\mathfrak{s}}^{\alpha-\alpha'} \cdot 2^{-j(\alpha'+\theta-|k|_{\mathfrak{s}})} \\
&\lesssim K\|y-x\|_{s}^{\alpha+\theta-|k|_{\mathfrak{s}}} .
\end{aligned}$$

It remains to estimate (again with Lemma 6.1.5)

$$
\begin{aligned}
\Big|\sum_{j>j'} \Gamma^{X^k}_{yx}\mathcal{J}_j(x)\tau\Big| &= \Big|\sum_{j>j'}\sum_{l\geqslant k\colon |l|_{\mathfrak{s}}<\alpha+\theta} \frac{1}{l!}\Gamma^{X^k}_{yx}X^l \cdot \int \Pi_x\tau(\mathrm{d}u)\,\partial^l\mathscr{A}^j_{x-u}\Big| \\
&\lesssim K\sum_{j>j'}\sum_{l\geqslant k\colon |l|_{\mathfrak{s}}<\alpha+\theta} \|y-x\|_{\mathfrak{s}}^{|l|_{\mathfrak{s}}-|k|_{\mathfrak{s}}}2^{-j(\alpha+\theta-|l|_{\mathfrak{s}})} \\
&\lesssim K\cdot\|y-x\|^{\alpha+\theta-|k|_{\mathfrak{s}}}\,.
\end{aligned}
$$

Since the regime $\|y-x\|_{\mathfrak{s}}>1$ can be treated by the same argumentes as the case $j>j'$ (with $j'=-1$), the proof for the bound on $\Gamma_{yx}\mathcal{I}\tau$ is closed.

We are left with the task to bound $\tilde{\Pi}_x\mathcal{I}\tau$ given by (6.14). Consider φ with $[\varphi]\leqslant 1$, where $[\varphi]$ is as in 2.3.10. We have to study for $\lambda\in(0,1]$

$$
\begin{aligned}
\tilde{\Pi}_x\mathcal{I}\tau(\varphi^\lambda_x) &= \sum_{j\geqslant 0}\int \mathrm{d}u\,\varphi^\lambda_{u-x}\Pi_x\tau(R^{\alpha+\theta}_{x-\cdot;u-x}\mathscr{A}^j) \\
&= \sum_{j\leqslant j'}\int \mathrm{d}u\,\varphi^\lambda_{u-x}\,\Pi_x\tau(R^{\alpha+\theta}_{x-\cdot;u-x}\mathscr{A}^j) - \sum_{j>j'}\int \mathrm{d}u\,\varphi^\lambda_{u-x}\,\Pi_x\tau(R^{\alpha+\theta}_{x-\cdot;u-x}\mathscr{A}^j)\,,
\end{aligned}
$$

where we picked $j'\geqslant 0$ such that $2^{-j'-1}\leqslant\lambda\leqslant 2^{-j'}$. We start by estimating the first term for which we use once more Lemma 2.1.20 and Lemma 6.1.5:

$$
\begin{aligned}
&\Big|\sum_{j\leqslant j'}\int \mathrm{d}u\,\varphi^\lambda_{u-x}\,\Pi_x\tau(R^{\alpha+\theta}_{x-\cdot;u-x}\mathscr{A}^j)\Big| \lesssim \sum_{j\leqslant j'}\sum_{\alpha'\colon\alpha'\leqslant\alpha}\sum_{k\in\mathbb{N}^d_{>\alpha+\theta}}\int \mathrm{d}u\,|\varphi^\lambda_{u-x}|\,\|u-x\|_{\mathfrak{s}}^{|k|_{\mathfrak{s}}} \\
&\times\int_0^1 \mathrm{d}t\,|\Pi_{x+v^k_t(u-x)}\Gamma^{\alpha'}_{x+v^k_t(u-x),x}\tau(\partial^k\mathscr{A}^j_{x+v^k_t(u-x)-\cdot})| \\
&\lesssim K\sum_{\alpha'\colon\alpha'\leqslant\alpha}\sum_{k\in\mathbb{N}^d_{>\alpha+\theta}}\int \mathrm{d}u\,|\varphi^\lambda_{u-x}|\,\|u-x\|^{|k|_{\mathfrak{s}}+\alpha-\alpha'}\sum_{j\leqslant j'}2^{-j(\alpha'+\theta-|k|_{\mathfrak{s}})} \\
&\lesssim K\sum_{\alpha'\colon\alpha'\leqslant\alpha}\sum_{k\in\mathbb{N}^d_{>\alpha+\theta}}\lambda^{\alpha-\alpha'+|k|_{\mathfrak{s}}}\sum_{j\leqslant j'}2^{-j(\alpha'+\theta-|k|_{\mathfrak{s}})} \lesssim K\cdot\lambda^{\alpha+\theta}\,,
\end{aligned}
$$

where we used that $\|v^k_t(u-x)\|_{\mathfrak{s}}\leqslant\|u-x\|_{\mathfrak{s}}$. The high frequency term can then be estimated by splitting $R^{\alpha+\theta}_{x-\cdot;u-x}\mathscr{A}^j=\mathscr{A}^j_{u-\cdot}-T^{\alpha+\theta}_{x-\cdot;u-x}\mathscr{A}^j$, so that we obtain with (6.6) and (6.7) from Lemma 6.1.5 the bound

$$
\begin{aligned}
&\sum_{j>j'}\Big|\int \mathrm{d}u\,\varphi^\lambda_{u-x}\,\Pi_x\tau(\mathscr{A}^j_{u-\cdot})\Big| + \sum_{j>j'}\Big|\int \mathrm{d}u\,\varphi^\lambda_{u-x}\,\Pi_x\tau(T^{\alpha+\theta}_{x-\cdot;u-x}\mathscr{A}^j)\Big| \\
&\lesssim K\sum_{j>j'}2^{-j\theta}\,\lambda^\alpha + K\sum_{j>j'}\sum_{k\in\mathbb{N}^d_{<\alpha+\theta}}\int \mathrm{d}u\,|\varphi^\lambda_{u-x}|\|u-x\|_{\mathfrak{s}}^{|k|_{\mathfrak{s}}}\,|\Pi_x\tau(\partial^k\mathscr{A}^j_{x-\cdot})| \\
&\lesssim K\Big(\lambda^\alpha\sum_{j>j'}2^{-j\theta} + \sum_{k\in\mathbb{N}^d_{<\alpha+\theta}}\lambda^{|k|_{\mathfrak{s}}}\cdot\sum_{j\gtrsim j'}2^{-j(\alpha+\theta-|k|_{\mathfrak{s}})}\Big) \lesssim K\cdot\lambda^{\alpha+\theta}
\end{aligned}
$$

This shows in particular that (6.10) is convergent and finishes the proof that $(\tilde{\Pi}, \tilde{\Gamma})$ is once more a model. (6.12) is practically only a repetition of the estimates above. □

By a successive application of Theorem 6.1.8 one can then build a regularity structure that is rich enough to solve the fixed point problem that is associated to a singular SPDE, compare [Hai14]. We will not pursue this point further, as our goal is to derive Schauder estimates for modelled distributions on an already constructed regularity structure. The following subsection will play a crucial role in this context.

6.1.2 Commutation with paraproducts

The following theorem is in a sense a broad generalization of [Tay00, Proposition 2.2] and of Lemma 4.1.7, but for space-time paraproducts instead. Similar as Lemma 4.1.7 was the key result that allowed for the proof of the a priori estimates in Section 4.3, Theorem 6.1.9 will be the essential ingredient for our Schauder estimates in Section 6.2 below.

Theorem 6.1.9. *Suppose we are given a regularity structure $\mathscr{T} = (A, \mathcal{T}, G)$ that satisfies Assumption 2.3.12 and a sector $\mathcal{V} \subseteq \mathcal{T}$. Suppose further we have a model (Π, Γ) that realizes the associated kernel $\mathscr{A}$ of some multiplier $a(D) \in \mathcal{M}_s^\theta(\mathbb{R}^d)$ of order $\theta \in \mathbb{R}$ for an abstract integration map $\mathcal{I}$ on $\mathcal{V}$.*

We then have for $\gamma \in \mathbb{R}$ and $F \in \mathcal{D}^\gamma(\mathbb{R}^d; \mathcal{V}\backslash\overline{\mathcal{T}}, \Gamma)$

$$\|\mathscr{A} * P(F, \Pi) - P(\mathcal{I}F, \Pi)\|_{\mathcal{C}_\mathfrak{s}^{\gamma+\theta}(\mathbb{R}^d)} \lesssim K_1 \cdot \|F\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\overline{\mathcal{T}})}, \tag{6.18}$$

$$\|P(F, \Pi) - a(D)P(\mathcal{I}F, \Pi)\|_{\mathcal{C}_\mathfrak{s}^{\gamma}(\mathbb{R}^d)} \lesssim K_1 \cdot \|F\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\overline{\mathcal{T}})}, \tag{6.19}$$

where $\mathscr{A} * P(F, \Pi)$ should be read as in Lemma 6.1.3 and where K_1 is some polynomial in $|\!|\!|(\Pi, \Gamma)|\!|\!|_\gamma$. Given a second model $(\hat{\Pi}, \hat{\Gamma})$ and some $\hat{F} \in \mathcal{D}^\gamma(\mathbb{R}^d; \mathcal{V}\backslash\overline{\mathcal{T}}, \hat{\Gamma})$ we further have

$$\begin{aligned}\|\mathscr{A} * P(F, \Pi) - P(\mathcal{I}F, \Pi) - (\mathscr{A} * P(\hat{F}, \hat{\Pi}) - P(\mathcal{I}\hat{F}, \hat{\Pi}))\|_{\mathcal{C}_\mathfrak{s}^{\gamma+\theta}(\mathbb{R}^d)} \\ \lesssim K_2 \cdot \left(\|F; \hat{F}\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\overline{\mathcal{T}},\Gamma,\hat{\Gamma})} + |\!|\!|(\Pi, \Gamma); (\hat{\Pi}, \hat{\Gamma})|\!|\!|_\gamma\right),\end{aligned} \tag{6.20}$$

$$\begin{aligned}\|P(F, \Pi) - a(D)P(\mathcal{I}F, \Pi) - (P(\hat{F}, \hat{\Pi}) - a(D)P(\mathcal{I}\hat{F}, \hat{\Pi}))\|_{\mathcal{C}_\mathfrak{s}^{\gamma}(\mathbb{R}^d)} \\ \lesssim K_2 \cdot \left(\|F; \hat{F}\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\overline{\mathcal{T}},\Gamma,\hat{\Gamma})} + |\!|\!|(\Pi, \Gamma); (\hat{\Pi}, \hat{\Gamma})|\!|\!|_\gamma\right),\end{aligned} \tag{6.21}$$

where K_2 is a polynomial in the corresponding norms of $F, \hat{F}, (\Pi, \Gamma)$ and $(\hat{\Pi}, \hat{\Gamma})$.

Remark 6.1.10. *There is no comparable result for the paraproducts $P(F, \Gamma^\alpha)$. In fact, if $\alpha \notin |\mathbb{N}^d|_\mathfrak{s}$ one has instead*

$$P(\mathcal{I}F, \Gamma^\alpha) = \mathcal{I}P(F, \Gamma^{\alpha-\theta})$$

by definition of an abstract integration map.

Proof. It is enough to prove (6.18) and (6.20), since (6.19) and (6.21) then follow from Lemma 6.1.3 and the fact that $a(D)\,(\mathscr{A} * P(F,\Pi)) - P(F,\Pi) = -\Psi^{-1} * P(F,\Pi) \in C^\infty(\mathbb{R}^d)$. We show (6.18). The distance estimate (6.20) follows by the same arguments.

Using that for u,v the map $w \mapsto \Psi^{<j-1}_{w-u}\Psi^j_{w-v}$ is spectrally supported on a rectangular annulus of size $2^{j\mathfrak{s}}$ we can write, using Lemma 2.1.12, for some $N > 0$ and $\tilde{\mathscr{A}^j} := \sum_{i:\,|i-j|\leqslant N} \mathscr{A}^j$

$$(\mathscr{A} * P(F,\Pi))_x = \sum_{j>0} \iiint \mathrm{d}w\,\mathrm{d}u\,\Pi_u F_u(\mathrm{d}v)\;\tilde{\mathscr{A}}^j_{x-w}\,\Psi^{<j-1}_{w-u}\Psi^j_{w-v}\,.$$

Using Lemma 2.1.14 to delete the polynomial contribution coming from (6.9) we can further write

$$\begin{aligned} P(\mathcal{I}F,\Pi)_x &= \sum_{j>0} \iiint \mathrm{d}w\,\mathrm{d}u\,\Pi_u F_u(\mathrm{d}v)\;\Psi^{<j-1}_{x-u}\Psi^j_{x-w}\mathscr{A}_{w-v} \\ &= \sum_{j>0} \iiint \mathrm{d}w\,\mathrm{d}u\,\Pi_u F_u(\mathrm{d}v)\;\Psi^{<j-1}_{x-u}\Psi^j_{x-w}\tilde{\mathscr{A}}^j_{w-v} \\ &= \sum_{j>0} \iiint \mathrm{d}w\,\mathrm{d}u\,\Pi_u F_u(\mathrm{d}v)\;\Psi^{<j-1}_{x-u}\Psi^j_{w-v}\tilde{\mathscr{A}}^j_{x-w}\,, \end{aligned}$$

where we used spectral support properties of Ψ^j to restrict us to the same $\tilde{\mathscr{A}^j}$ as above and where we substituted $w := x+v-w$ in the integral in w in the third line (to justify the last step for distributions one can argue by approximation). Altogether we obtain with $K^j_x(u,v) := (\Psi^{<j-1}_{w-u} - \Psi^{<j-1}_{x-u})\tilde{\mathscr{A}}^j_{x-w}$

$$\begin{aligned} &(\mathscr{A} * P(F,\Pi) - P(\mathcal{I}F,\Pi))_x = \sum_{j>0} \iiint \mathrm{d}w\,\mathrm{d}u\,\Pi_u F_u(\mathrm{d}v)\;K^j_x(u,w)\,\Psi^j_{w-v} \\ &= \sum_{j>0} \iiint \mathrm{d}w\,\mathrm{d}u\,\Pi_u(F_u - \Gamma_{ux}F_x)(\mathrm{d}v)\;K^j_x(u,w)\,\Psi^j_{w-v} \\ &= \sum_{j>0}\sum_{\alpha\in A_{\mathcal{V}\setminus\overline{\mathcal{T}}}}\;\sum_{\alpha'\in A_{\mathcal{V}\setminus\overline{\mathcal{T}}}:\,\alpha'\leqslant\alpha} \iint \mathrm{d}w\,\mathrm{d}u\,K^j_x(u,w)\cdot\Pi_w\big(\Gamma^{\alpha'}_{wu}(F^\alpha_u - \Gamma^\alpha_{ux}F_x)\big)(\Psi^j_{w-\cdot})\,, \end{aligned} \tag{6.22}$$

where we smuggled in a term $\Pi_x F_x(\Psi^j_{w-\cdot})$ in the second line, which vanishes when integrated against $K^j_x(u,w)$ over u, and used $\Pi_u = \Pi_w\Gamma_{wu}$ in the last line. We further erased all α' belonging to polynomial components as they vanish when integrated against Ψ^j due to Lemma 2.1.14. Because we did not include $\alpha' \in A_{\overline{\mathcal{T}}}$ in the summation we can also neglect the cases $\alpha \in A_{\overline{\mathcal{T}}}$ since $\overline{\mathcal{T}}$ is a sector.

Every term of the sum in (6.22) is a difference of the corresponding j term of $\mathscr{A} * P(F,\Pi)$ and of $P(\mathcal{I}F,\Pi)$ and is therefore spectrally supported in a rectangular annulus scaled by $2^{j\mathfrak{s}}$. Thus, by Lemma 2.1.19, it is enough to bound each of these terms. Using Lemma 2.3.10 and the fact that $F \in \mathcal{D}^\gamma(\mathbb{R}^d; \mathcal{V}\backslash\overline{\mathcal{T}})$ we can bound each term by

$$\begin{aligned}
&K_1 \cdot \|F\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\overline{\mathcal{T}})} \sum_{\alpha \in A_{\mathcal{V}\backslash\overline{\mathcal{T}}}} \sum_{\alpha' \in A_{\mathcal{V}\backslash\overline{\mathcal{T}}}\colon \alpha' \leqslant \alpha} 2^{-j\alpha'} \\
&\times \iint \mathrm{d}w\mathrm{d}u \, |K_x^j(u,w)| \, \|u-w\|_{\mathfrak{s}}^{\alpha-\alpha'} \, \|u-x\|_{\mathfrak{s}}^{\gamma-\alpha} \\
&\lesssim K_1 \|F\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\overline{\mathcal{T}})} \sum_{\alpha \in A_{\mathcal{V}\backslash\overline{\mathcal{T}}}} \sum_{\alpha' \in A_{\mathcal{V}\backslash\overline{\mathcal{T}}}\colon \alpha' \leqslant \alpha} 2^{-j\alpha'} \\
&\Big(\iint \mathrm{d}w\mathrm{d}u \, |\Psi_{w-u}^{<j-1} \tilde{\mathscr{A}}_{x-w}^j| \, \|u-w\|_{\mathfrak{s}}^{\alpha-\alpha'} \, \|u-x\|_{\mathfrak{s}}^{\gamma-\alpha} \\
&+ \iint \mathrm{d}w\mathrm{d}u \, |\Psi_{x-u}^{<j-1} \tilde{\mathscr{A}}_{x-w}^j| \, \|u-w\|_{\mathfrak{s}}^{\alpha-\alpha'} \, \|u-x\|_{\mathfrak{s}}^{\gamma-\alpha} \Big) \\
&\lesssim K_1 \|F\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\overline{\mathcal{T}})} \sum_{\alpha \in A_{\mathcal{V}\backslash\overline{\mathcal{T}}}} \sum_{\alpha' \in A_{\mathcal{V}\backslash\overline{\mathcal{T}}}\colon \alpha' \leqslant \alpha} 2^{-j\alpha'} \big(2^{-j(\alpha-\alpha'+\gamma-\alpha+\theta)} \big) \\
&\lesssim K_1 \|F\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\overline{\mathcal{T}})} 2^{-j(\gamma+\theta)} ,
\end{aligned}$$

where we used Lemma 2.1.14 together with Lemma 6.1.4 and the inequalities

$$\begin{aligned}
\|u-x\|_{\mathfrak{s}}^{\gamma-\alpha} &\lesssim \|u-w\|_{\mathfrak{s}}^{\gamma-\alpha} + \|w-x\|_{\mathfrak{s}}^{\gamma-\alpha} , \\
\|u-w\|_{\mathfrak{s}}^{\alpha-\alpha'} &\lesssim \|u-x\|_{\mathfrak{s}}^{\alpha-\alpha'} + \|x-w\|_{\mathfrak{s}}^{\alpha-\alpha'} .
\end{aligned}$$

Application of Lemma 2.1.19 finishes the proof. □

6.2 Schauder estimates

We fix a regularity structure $\mathcal{T}$ that satisfies Assumption 2.3.12 together with a model (Π, Γ) and some scaling vector $\mathfrak{s} \in [1, \infty)^d$ in the form

$$\mathfrak{s} = (\theta, 1, \ldots, 1) ,$$

where $\theta > 1$ is assumed for simplicity to be an integer, compare Remark 6.2.1 below. In this section we give Schauder estimates to an operator

$$a(D) = \partial_t - p(D') \in \mathcal{M}_{\mathfrak{s}}^\theta(\mathbb{R}^d) \tag{6.23}$$

with p being some real-valued, homogeneous polynomial of order θ on $\mathbb{R}^{d-1}$ which is "elliptic", in the sense that

$$p(z) \lesssim -|z|^\theta ,$$

so that θ is in fact an *even* integer. In particular $p \in \mathcal{M}^{\theta}_{(1,\dots,1)}(\mathbb{R}^{d-1}) \cap C^{\infty}(\mathbb{R}^{d-1})$. Since θ is an integer in this section we now have in particular

$$|\mathbb{N}^d|_{\mathfrak{s}} = \mathbb{N}\,, \qquad A_{\mathbb{N}^d} = A + \mathbb{N}\,.$$

Remark 6.2.1 (Non-integer values for θ)**.** *We need θ to be an integer in order to assure that $a(D)$ as constructed in* (6.23) *is really a smooth multiplier in the sense of Definition 6.1.1. If $\theta > 1$ is fractional, $p \in \mathcal{M}^{\theta}_{(1,\dots,1)}(\mathbb{R}^{d-1})$ is typically not (infinitely) differentiable in $0 \in \mathbb{R}^{d-1}$, so that a as given by* (6.23) *is not (infinitely) differentiable at the subspace $\{x = (x_1, x') \in \mathbb{R}^d \,|\, x' = 0\}$. However, we only need smoothness in order to assure that*

$$\sup_{x\in\mathbb{R}^d} \|x\|_{\mathfrak{s}}^N |\partial^k \mathscr{A}_0(x)| < \infty \tag{6.24}$$

with $\mathscr{A}_0$ as in Lemma 6.1.4 and N big enough such that the proof of Lemma 6.1.5 works (If one considers $\alpha \in A$ with $\alpha_0 \leqslant \alpha < \gamma$, the choice $N = |\mathfrak{s}| + \gamma - \alpha_0$ will do), so that one might allow for fractional θ if these θ still allow to get (6.24)*.*

Integration maps

For functions $f : \mathbb{R}\times\mathbb{R}^{d-1} \to \mathbb{C}$, for which this is defined, we introduce the operations

$$If(t, x') = \int_{-\infty}^{t} \left(e^{(t-s)p(D')} f(s, \cdot)\right)(x')\,\mathrm{d}s\,, \tag{6.25}$$

$$I_0 f(t, x') := \int_{0}^{t} \left(e^{(t-s)p(D')} f(s, \cdot)\right)(x')\,\mathrm{d}s\,. \tag{6.26}$$

The operator $e^{tp(D')}$ can be seen as some natural generalization of the heat semigroup $e^{t\Delta}$ and it essentially interacts in the same way with distributions (compare Lemma 6.3.2 below). If one writes $G(x_1, x') := \mathbf{1}_{x_1>0}\mathcal{F}_{\mathbb{R}^{d-1}}(e^{x_1 p(\cdot)})(x')$ for $x = (x_1, x') \in \mathbb{R}^d$ one checks that $G *_{\mathbb{R}^d} \phi = \mathscr{A} * \phi$ for $\phi \in \mathcal{S}(\mathbb{R}^d)$ with $\operatorname{supp}\mathcal{F}_{\mathbb{R}^d}\phi \subseteq B^c := (\operatorname{supp}\varphi_{-1})^c$. Consequently one can define for *distributional* $f \in \mathcal{S}'(\mathbb{R}^d)$

$$If := I(\Psi^{<1} * f) + \mathscr{A} * \tilde{f}\,, \tag{6.27}$$

where $\tilde{f} = f - \Psi^{<1} * f = \sum_{j\geqslant 1} \varphi_j(D) f$ has spectral support on B^c, compare the properties of our dyadic partition of unity on page 29. Of course we can only define If provided that the two terms in (6.27) are well-defined. For $\beta \in \mathbb{R}$ and $f \in \mathcal{C}^{\beta}_{\mathfrak{s}}(\mathbb{R}^d)$ the second term of (6.27) is well-defined in the space $\mathcal{C}^{\beta+\theta}_{\mathfrak{s}}(\mathbb{R}^d)$ by Lemma 6.1.3 if one uses

$$f - \tilde{f} = \Psi^{<1} * f \in \mathcal{C}^{\infty}_{\mathfrak{s}}(\mathbb{R}^d) = \bigcap_{\gamma\in\mathbb{R}} \mathcal{C}^{\gamma}_{\mathfrak{s}}(\mathbb{R}^d)\,, \tag{6.28}$$

(which follows from Lemma 2.1.19). If f has compact support in time, say $\operatorname{supp} f \subseteq [0,T] \times \mathbb{R}^{d-1}$, one sees that (6.28) decays faster than any polynomial in its time variable, so that the first term in (6.27) is well-defined by the representation (6.25) and our choice of p and one obtains once more

$$I(\Psi^{<1} * f) \in \mathcal{C}^{\infty}_{\mathfrak{s}}(\mathbb{R}^d)\,. \tag{6.29}$$

For such f, that is with support in $[0,T] \times \mathbb{R}^{d-1}$, we will further write

$$I_0 f = I f$$

in accordance with (6.26). By approximation one checks that $\operatorname{supp} I_0 f \subseteq \mathbb{R}_+ \times \mathbb{R}^{d-1}$. Summarizing the findings above we have for $f \in \mathcal{C}^{\beta}(\mathbb{R}^d)$ with support in $[0,T] \times \mathbb{R}^{d-1}$

$$\|I_0 f\|_{\mathcal{C}^{\beta+\theta}} \lesssim \|f\|_{\mathcal{C}^{\beta}(\mathbb{R}^d)},\ \operatorname{supp}\, I_0 f \subseteq \mathbb{R}_+ \times \mathbb{R}^{d-1}\,. \tag{6.30}$$

Preliminaries

In the following we fix parameters $\alpha \in A$, $\eta \in \mathbb{R}\backslash A_{\mathbb{N}^d}$, $\tilde{\eta}, \overline{\eta}$, γ, $\overline{\gamma} \in (0,\infty)\backslash A_{\mathbb{N}^d}$, and $\kappa > 0$ (this choice is in a way inspired by [Hai14, Subsection 7.3]) such that

$$\begin{aligned} -\theta &< \alpha \wedge \eta\,, & \eta &< \gamma\,,\\ \overline{\eta} &< \eta \wedge \alpha + \theta\,, & \overline{\gamma} &< \gamma + \theta\,,\\ \tilde{\eta} &= \overline{\eta} - \kappa\,, & \tilde{\eta} - \overline{\gamma} &> -\theta\,. \end{aligned}$$

We also assume $\gamma + \theta$, $\eta + \theta \notin A_{\mathbb{N}^d}$. One should think of the parameters $\overline{\eta}, \tilde{\eta}$ and $\overline{\gamma}$ as being "almost identical" to $\eta + \theta$ and $\gamma + \theta$ respectively.

Let $\mathcal{V} \subseteq \mathcal{T}$ of regularity α such that $(\mathcal{V}, \overline{\mathcal{T}})$ is $\overline{\gamma}$-regular. Assume we are given an abstract integration map $\mathcal{I}$ on $\mathcal{V}$ such that the model (Π, Γ) realized the associated kernel $\mathscr{A} = \sum_{j \geqslant 0} \mathscr{A}^j$ of (6.23) for $\mathcal{I}$.

Our aim is to build a continuous mapping

$$\mathcal{K}:\ \mathcal{D}^{[\eta,\gamma]}(\Omega^T; \mathcal{V}) \to \mathcal{D}^{[\tilde{\eta},\overline{\gamma}]}(\Omega^T; \mathcal{T})\,,$$

such that $\mathcal{RK}F = I\mathcal{R}F$ and $\mathcal{K}F|_{\mathcal{T}\backslash\overline{\mathcal{T}}} = \mathcal{I}F$, where $\mathcal{R}$ is the reconstruction operator from Lemma 5.3.11. Note, that we used that $\mathcal{R}F$ has support in $[0,T] \times \mathbb{R}^{d-1}$, since we extended F from $[0,T] \times \mathbb{R}^{d-1}$ by 0. Consequently, $I\mathcal{R}F = I_0\mathcal{R}F$ is well-defined and one further sees by (6.30) and our choice of parameters that $I\mathcal{R}F$ is in fact a *function.*

We will construct the map $\mathcal{K}$ as a sum

$$\mathcal{K}F = \mathcal{I}F + \mathcal{P}F\,,$$

where we will choose $\mathcal{P}F \in \overline{\mathcal{T}}$. Since $\mathcal{K}F$ will live in a function-like sector by choice of our parameters above (compare Remark 6.2.2 below), we must have $\mathcal{P}^{\mathbf{1}}F = I\mathcal{R}F \in \mathcal{C}_{\mathfrak{s}}^{\overline{\gamma}}(\Omega^T)$ (for the notation $(\ldots)^{\mathbf{1}}$ compare page 46). The structure condition (5.13) (considering $\alpha = 0$) leaves us then but one choice for $\mathcal{P}$, which is for $x \in \Omega^T$ and $\mu \in \mathbb{N}^d$ with $0 \leqslant |\mu|_{\mathfrak{s}} < \overline{\gamma}$

$$\mathcal{P}_x^{X^\mu} F = (\mathcal{P}^{X^\mu} F)(x) = \frac{1}{\mu!} \lim_{N\to\infty} \int \mathrm{d}u\, \partial^\mu \Psi_{x-u}^{<N} \left(I\mathcal{R}F(u) - \Gamma_{ux}^{\mathbf{1}} \mathcal{I}(F_x) \right) . \qquad (6.31)$$

Recall that the notation $\mathcal{P}_x^{X^\mu}F$ means "the coefficient of $\mathcal{P}_x^{X^\mu}F$ in front of X^μ", compare page 46. Note that in particular $\mathcal{P}_x^{\mathbf{1}}(F) = \mathcal{P}_x^{X^0}(F) = (I\mathcal{R}F)_x - \Gamma_{xx}\mathcal{I}(F_x) = (I\mathcal{R}F)_x$, so that our choice is consistent. The fact that $\mathcal{P}^{X^\mu}$ is well-defined is of course non-trivial, and will be a byproduct of the proof of the main theorem of this section, Theorem 6.2.3 below.

Remark 6.2.2 (Are function-like sectors enough?)**.** *Recall that $\mathcal{V}$ has lowest regularity α and that by our assumption $\alpha + \theta > 0$. Consequently, the map $\mathcal{K}$ only takes values in some function-like sector, namely*

$$\mathcal{W} := \mathcal{I}(\mathcal{V}) + \overline{\mathcal{T}} .$$

The image $\mathcal{I}(\mathcal{V})$ of the sector $\mathcal{V}$ under $\mathcal{I}$ is then a complement of a sector, more precisely $\mathcal{W}\backslash\overline{\mathcal{T}}$, with regularity $\alpha + \theta > 0$.

The assumption on α might be a bit surprising since some singular SPDEs seem not to fit in this framework. Consider, for example, the (Φ_3^4)-model where the right hand side will be described by a modelled distribution of the form

$$-\Phi^{\star 3} + \Xi , \qquad (6.32)$$

here Φ is the modelled distribution that is thought to describe the solution to (Φ_3^4), while Ξ is a symbol that represents space-time white noise. Ξ will be of homogeneity $-5/2 - \varepsilon$ for any $\varepsilon > 0$ and since $\theta = 2$ for (Φ_3^4) we see that (6.32) does not *take values in a sector that satisfies the assumptions above. However, we can construct the symbol $\mathcal{I}(\Xi)$ "by hand" because it simply represents the integration of noise against the Green's function of the heat operator. We can then* define *the integrated version of (6.32) as*

$$-\mathcal{K}(\Phi^{\star 3}) + \mathcal{I}(\Xi) ,$$

which makes sense as one can check that $\Phi^{\star 3}$ takes values in a sector that satisfies the assumptions above. By this proceeding one can derive Schauder estimates and solve the equation.

A similar remark applies for other singular SPDEs as well [Hai14, Remark 7.9], so that the requirements above are in fact not a big restriction.

Schauder estimates

The proof of our Schauder estimates will essentially be an application of three properties for the operator $a(D)$ and its "inverse "$\mathscr{A}$:

1. $a(D)$ acts locally in time.
2. $a(D)$ commutes with paraproducts as in Theorem 6.1.9.
3. The integration against $\mathscr{A}$ has some suitable "classical" Schauder estimates.

From this point of view our approach is much closer to the one in [GIP15, GP15b] or in Chapter 4 and not really related to the proceeding in [Hai14] (with the exception of the "easy" estimates on the abstract integration map $\mathcal{I}$). Point 3 was expressed in a fuzzy way, since there are probably quite a few ways in which such estimates can be stated. We here use that we have for the parameters above

$$\sup_{t\in(0,T)} \sup_{\beta\in[\bar{\eta},\bar{\gamma}]\setminus A_{\mathbb{N}^d}} t^{\frac{\beta-\tilde{\eta}}{\theta}} \|I_0 f\|_{\mathcal{C}^{\beta}_{\mathfrak{s}}(\Omega^T_t)} \lesssim \sup_{t\in(0,T]} \sup_{\beta\in([\eta,\gamma]\setminus A_{\mathbb{N}^d})\cap(0,\gamma]} t^{\frac{\beta-\eta}{\theta}} \|f\|_{\mathcal{C}^{\beta}_{\mathfrak{s}}(\Omega^T_t)} ,$$

which is a consequence of the estimates on the semigroup $e^{tp(D')}$. We give a proof for this statement in Lemma 6.3.3 below. For point 2 the key role will be played by Theorem 6.1.9, which states that (with F suitably extended to $\mathbb{R}^d$) $a(D)P(\mathcal{I}(F),\Pi) - P(F,\Pi)$ is smooth to some extent. Since $\mathcal{I}(F)$ takes values in a function-like sector we have $\Pi_x\mathcal{I}(F) = \Gamma^{\mathbf{1}}_{\cdot x}\mathcal{I}(F) = (\Gamma_{\cdot x}\mathcal{I}(F))^{\mathbf{1}}$, which is a "well-known" result which we recall below (Lemma 6.3.6), so that in particular

$$P(\mathcal{I}(F),\Pi) = P(\mathcal{I}(F),\Gamma^{\mathbf{1}}) ,$$

where $P(\mathcal{I}(F),\Gamma^{\mathbf{1}})$ is defined as in Remark 5.1.2. This identity will allow us to switch between the different paraproducts introduced in Definition 5.1.1. Based on these techniques we prove the following Schauder estimate.

Theorem 6.2.3. *Consider the setup above. The function $\mathcal{P}$, defined by* (6.31), *is well-defined for $F \in \mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V})$, and $\mathcal{K}F := \mathcal{P}(F) + \mathcal{I}(F)$ is a map $\mathcal{K} : \mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}) \to \mathcal{D}^{[\bar{\eta},\bar{\gamma}]}(\Omega^T;\mathcal{T})$ that satisfies the bounds*

$$\mathcal{R}\mathcal{K}F = I\mathcal{R}F , \tag{6.33}$$

$$\|\mathcal{K}F\|_{\mathcal{D}^{[\bar{\eta},\bar{\gamma}]}(\Omega^T;\mathcal{V})} \lesssim K_1 \cdot T^{\frac{\kappa}{2\theta}} \|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{T})} , \tag{6.34}$$

where K_1 is a polynomial in $|||(\Pi,\Gamma)|||_{\bar{\gamma}}$. Given a second model $(\hat{\Pi},\hat{\Gamma})$ define $\hat{\mathcal{K}}$ in the exact same manner. We then have

$$\|\mathcal{K}F;\hat{\mathcal{K}}\hat{F}\|_{\mathcal{D}^{[\bar{\eta},\bar{\gamma}]}(\Omega^T;\mathcal{V},\Gamma,\hat{\Gamma})} \lesssim K_2 \cdot T^{\frac{\kappa}{2\theta}} \left(\|F;\hat{F}\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{T},\Gamma,\hat{\Gamma})} + |||(\Pi,\Gamma);(\hat{\Pi},\hat{\Gamma})|||_{\bar{\gamma}}\right) , \tag{6.35}$$

where K_2 is some polynomial in the corresponding norms of $F, \hat{F}, (\Pi,\Gamma), (\hat{\Pi},\hat{\Gamma})$.

Proof. Note that once we can prove $\mathcal{K}F \in \mathcal{D}^{[\tilde{\eta},\overline{\gamma}]}(\mathbb{R}^d;\mathcal{T})$, (6.33) follows by [Hai14, Proposition 3.28]. Our task is therefore reduced to show (6.34). By doing so it will also turn out that (6.31) is convergent, so that $\mathcal{P}^{X^\mu}F$ is well-defined.

We will include the polynomial K_1, which will change from line to line, in the notation "$\lesssim$" to shorten the formulas a bit.

We introduce the sector $\mathcal{W} := \mathcal{I}(\mathcal{V}) + \overline{\mathcal{T}}$ in which $\mathcal{K}F$ takes values, due to the properties of an abstract integration map. Note that by our assumption on α the regularity of $\mathcal{W}$ is 0 and note further that $\mathcal{W}\backslash\overline{\mathcal{T}} = \mathcal{I}(\mathcal{V})$. In particular $\mathcal{W}$ is function-like and for $\tau \in \mathcal{W}$ we have $\Pi_x\tau = \Gamma^{\mathbf{1}}_{\cdot x}\tau$ by Lemma 6.3.6 below. For $\tau \in \mathcal{W}$ and $B \in \{\overline{\mathcal{T}}, \mathcal{W}\backslash\overline{\mathcal{T}}\}$ we write in this proof shorthand τ^B for the projection of τ on the subspace B and set $\Gamma^B\tau := (\Gamma\tau)^B$.

Our proof of (6.34) will split in several parts.

(a) For $\kappa' \geqslant 0$, $\overline{\eta}_{\kappa'} := \overline{\eta} - \kappa' \in (0,\infty)\backslash A_{\mathbb{N}^d}$ we have $\mathcal{I}(F) \in \mathcal{D}^{[\overline{\eta}_{\kappa'},\overline{\gamma}]}(\Omega^T;\mathcal{W}\backslash\overline{\mathcal{T}})$ with $\|\mathcal{I}(F)\|_{\mathcal{D}^{[\overline{\eta}_{\kappa'},\overline{\gamma}]}(\Omega^T;\mathcal{W}\backslash\overline{\mathcal{T}})} \lesssim T^{\frac{\kappa'}{\theta}}\|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V})}$, in particular

$$\|\mathcal{I}(F)\|_{\mathcal{D}^{[\tilde{\eta},\overline{\gamma}]}(\Omega^T;\mathcal{W}\backslash\overline{\mathcal{T}})} \lesssim T^{\frac{\kappa}{\theta}}\|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V})}\,,$$

(b) For $t \in (0,T)$ and $\beta \in [\tilde{\eta},\overline{\gamma}]\backslash A_{\mathbb{N}^d}$ we have

$$t^{\frac{\beta-\tilde{\eta}}{\theta}}\|\mathcal{P}^{\mathbf{1}}F - P(\overline{\mathcal{I}(F^{<\beta})},\Pi)\|_{\mathcal{C}^{\beta}_{\mathfrak{s}}(\Omega^T_t)} \lesssim T^{\frac{\kappa}{2\theta}}\|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V})}$$

where $\overline{(\ldots)}$ denotes the "poor man's extension" from Subsection 5.3.2.

(c) We have for $t \in (0,T)$, $\beta \in [\tilde{\eta},\overline{\gamma}]\backslash A_{\mathbb{N}^d}$ and $\mu \in \mathbb{N}^d$ with $0 \leqslant |\mu|_{\mathfrak{s}} < \beta$ the bounds

$$t^{\frac{\beta-\tilde{\eta}}{\theta}}\left\|\mathcal{P}^{X^\mu}F - P\big(\mathcal{I}(\mathcal{E}_{\Omega^T_t}F^{<\beta-\theta}),\Gamma^{X^\mu}\big)\right\|_{\mathcal{C}^{\beta-|\mu|_{\mathfrak{s}}}_{\mathfrak{s}}(\Omega^T_t)} \lesssim T^{\frac{\kappa}{2\theta}}\|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V})}\,,$$
$$t^{\frac{\beta-\tilde{\eta}}{\theta}}\left\|\mathcal{P}^{X^\mu}F\right\|_{C_b(\Omega^T_t)} \lesssim T^{\frac{\kappa}{2\theta}}\|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V})}$$

where $\mathcal{E}_{\Omega^T_t}$ denotes the Whitney extension from Theorem 5.3.16. In particular the operator $\mathcal{P}$ is well-defined.

(d) $\mathcal{K}F = \mathcal{P}F + \mathcal{I}F$ fulfills the structure condition (5.13) on Ω^T for $\alpha' \in A \cap |\mathbb{N}^d|_{\mathfrak{s}}$ and $\overline{\gamma}$.

Once (a),(c) and (d) are established, the claim is just an application of the local embedding property in Theorem 5.2.1. Indeed: Fix $\beta \in [\tilde{\eta},\overline{\gamma}]$ and $t \in (0,1)$ and observe that $\mathcal{I}(\mathcal{E}_{\Omega^T_t}F^{<\beta-\theta})|_{\Omega^T_t} = \mathcal{I}(F^{<\beta-\theta})|_{\Omega^T_t} = \mathcal{I}^{<\beta}(F)|_{\Omega^T_t}$, so that (a),(c) and (d) imply via Theorem 5.2.1

$$t^{\frac{\beta-\tilde{\eta}}{\theta}}\|\mathcal{K}^{<\beta}(F)\|_{\mathcal{D}^{\beta}(\Omega^T_t;\mathcal{T})} = t^{\frac{\beta-\tilde{\eta}}{\theta}}\|\mathcal{I}^{<\beta}(F) + \mathcal{P}^{<\beta}(F)\|_{\mathcal{D}^{\beta}(\Omega^T_t;\mathcal{T})} \lesssim T^{\frac{\kappa}{2\theta}}\|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V})}\,,$$

Taking finally the supremum over $t \in (0,T)$ shows (6.34). We need Part (b) to prove (c) and (d).

Part (a) is almost an immediate consequence of the definition of an abstract integration map and Lemma 5.3.8. Indeed, we have for $x, y \in \Omega_t^T$, $t \in (0,T)$ for some $\overline{\tau}(x,y) \in \overline{\mathcal{T}}$

$$\mathcal{I}F_y - \Gamma_{yx}\mathcal{I}F_x = \mathcal{I}\left(F_y - \Gamma_{yx}F_x\right) + \overline{\tau}(x,y) \tag{6.36}$$

so that we obtain with Lemma 5.3.8 and the continuity of $\mathcal{I}$ Part (a) of the proof

$$\|\mathcal{I}F\|_{\mathcal{D}^{[\overline{\eta}_{\kappa'},\overline{\gamma}]}(\Omega^T;\mathcal{W}\backslash\overline{\mathcal{T}})} \lesssim T^{\frac{\eta\wedge\alpha+\theta-\overline{\eta}_{\kappa'}}{\theta}}\|\mathcal{I}F\|_{\mathcal{D}^{[\eta+\theta,\gamma+\theta]}(\Omega^T;\mathcal{W}\backslash\overline{\mathcal{T}})} \lesssim T^{\frac{\kappa'}{\theta}}\|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V})}\,, \tag{6.37}$$

where we used in the last step $\overline{\eta} < \eta\wedge\alpha+\theta$ and $\overline{\eta}_{\kappa'} = \overline{\eta}-\kappa'$. We first show Part (b) for $\beta \in [\tilde{\eta},\overline{\eta}]\backslash A_{\mathbb{N}^d}$, recall that

$$\mathcal{K}^{\mathbf{1}}F = \mathcal{P}^{\mathbf{1}}F = I\mathcal{R}F$$

We know from Lemma 5.3.11 that for $\varepsilon > 0$ one has $\|\mathcal{R}F\|_{\mathcal{C}_{\mathfrak{s}}^{\eta\wedge\alpha-\varepsilon}(\mathbb{R}^d)} \lesssim \|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T)}$, $\operatorname{supp}\mathcal{R}F \subseteq [0,T]\times\mathbb{R}^{d-1}$ and with (6.30) we obtain $\|I\mathcal{R}F\|_{\mathcal{C}_{\mathfrak{s}}^{\overline{\eta}}(\mathbb{R}^d)} \lesssim \|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T)}$ with $I\mathcal{R}F(x) = 0$ whenever $x_1 < 0$. Recall that we chose $\overline{\eta} > 0$ and $\tilde{\eta} = \overline{\eta}-\kappa > 0$, so that a simple interpolation argument yields $t^{\frac{\beta-\tilde{\eta}}{\theta}}\|I\mathcal{R}F\|_{\mathcal{C}_{\mathfrak{s}}^{\beta}(\Omega_t^T)} \lesssim T^{\frac{\kappa}{\theta}}\|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T)}$. Since $F^{<\beta}$ only has polynomial entries for $\beta \in [\tilde{\eta},\overline{\eta}]$ so does $\overline{F^{<\beta}}$. Consequently $P(\overline{F^{<\beta}},\Pi) = 0$, so that (b) is showed for $\beta \in [\tilde{\eta},\overline{\eta}]$. For (b) with $\beta \in (\overline{\eta},\overline{\gamma}]\backslash A_{\mathbb{N}^d}$ we write

$$\begin{aligned} &I\mathcal{R}F - P(\overline{\mathcal{I}(F^{<\beta})},\Pi) \\ &= I_0\mathcal{R}F - I_0 a(D)P(\overline{\mathcal{I}(F^{<\beta})},\Pi) - e^{tp(D')}\left(P(\overline{\mathcal{I}(F^{<\beta})},\Pi)|_{\{x\,|x_1=0\}}\right)\,, \end{aligned} \tag{6.38}$$

where we used Duhamel's principle and the fact that $P(\overline{\mathcal{I}(F^{<\beta})},\Pi)$ "solves" the trivial problem

$$\begin{aligned} &a(D)P(\overline{\mathcal{I}(F^{<\beta})},\Pi) = a(D)P(\overline{\mathcal{I}(F^{<\beta})},\Pi), \quad P(\overline{\mathcal{I}(F^{<\beta})},\Pi)|_{\{x\,|x_1=0\}} \\ &= P(\overline{\mathcal{I}(F^{<\beta})},\Pi)|_{\{x\,|x_1=0\}}\,. \end{aligned}$$

The last term of (6.38) can be estimated with Lemma 6.3.7 below (recall that α was the regularity of $\mathcal{V}$)

$$\begin{aligned} &\left\|e^{tp(D')}\left(P(\overline{\mathcal{I}(F^{<\beta})},\Pi)|_{\{x\,|x_1=0\}}\right)\right\|_{\mathcal{C}_{\mathfrak{s}}^{\beta}(\Omega_t^T)} \lesssim t^{\frac{(\alpha+\theta)\wedge(\overline{\eta}_{\kappa/2}-\kappa/2)-\beta}{\theta}}\|\mathcal{I}(F)\|_{\mathcal{D}^{[\overline{\eta}_{\kappa/2},\overline{\gamma}]}(\Omega^T;\mathcal{W}\backslash\overline{\mathcal{T}})} \\ &= t^{\frac{(\alpha+\theta)\wedge\tilde{\eta}-\beta}{\theta}}\|\mathcal{I}(F)\|_{\mathcal{D}^{[\overline{\eta}_{\kappa/2},\overline{\gamma}]}(\Omega^T;\mathcal{W}\backslash\overline{\mathcal{T}})} \overset{\alpha+\theta>\overline{\eta}\geq\tilde{\eta}}{=} t^{\frac{\tilde{\eta}-\beta}{\theta}}\|\mathcal{I}(F)\|_{\mathcal{D}^{[\overline{\eta}_{\kappa/2},\overline{\gamma}]}(\Omega^T;\mathcal{W}\backslash\overline{\mathcal{T}})} \\ &\overset{(a)}{\lesssim} t^{\frac{\tilde{\eta}-\beta}{\theta}}\,T^{\frac{\kappa}{2\theta}}\,\|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V})}\,. \end{aligned}$$

The first two terms of (6.38) can be estimated via the "classical" Schauder estimate in Lemma 6.3.3 below (with $\overline{\eta}' = \overline{\eta}_{\kappa/2}$ therein):

$$\begin{aligned}
&\sup_{t\in(0,T)} \sup_{\beta\in[\overline{\eta},\overline{\gamma}]\backslash A_{\mathbb{N}^d}} t^{\frac{\beta-\bar{\eta}}{\theta}} \left\| I_0 \left(\mathcal{R}F - a(D)P(\overline{\mathcal{I}(F^{<\beta})}, \Pi) \right) \right\|_{\mathcal{C}^\beta_{\mathfrak{s}}(\Omega^T_t)} \\
&\leqslant T^{\frac{\kappa}{2\theta}} \sup_{t\in(0,T)} \sup_{\beta\in[\overline{\eta},\overline{\gamma}]\backslash A_{\mathbb{N}^d}} t^{\frac{\beta-\overline{\eta}_{\kappa/2}}{\theta}} \left\| I_0 \left(\mathcal{R}F - a(D)P(\overline{\mathcal{I}(F^{<\beta})}, \Pi) \right) \right\|_{\mathcal{C}^\beta_{\mathfrak{s}}(\Omega^T_t)} \\
&\overset{\text{Lem. 6.3.3}}{\lesssim} T^{\frac{\kappa}{2\theta}} \sup_{t\in(0,T)} \sup_{\beta\in([\eta,\gamma]\backslash A_{\mathbb{N}^d})\cap(0,\gamma]} t^{\frac{\beta-\eta}{\theta}} \| \mathcal{R}F - a(D)P(\overline{\mathcal{I}(F^{<\beta})}, \Pi) \|_{\mathcal{C}^\beta_{\mathfrak{s}}(\Omega^T_t)} \,.
\end{aligned}$$

But now we can apply Lemma 5.1.8 and Proposition 5.3.13 for $\tilde{F}^{<\beta} := \mathcal{E}_{\Omega^T_t} F^{<\beta}$ and $\mathcal{I}(\tilde{F}^{<\beta})$, along with Theorem 6.1.9 and Lemma 6.1.3 to estimate for $\beta \in ([\eta,\gamma]\backslash A_{\mathbb{N}^d}) \cap (0,\gamma]$

$$\begin{aligned}
&\| \mathcal{R}F - a(D)P(\overline{\mathcal{I}(F^{<\beta})}, \Pi) \|_{\mathcal{C}^\beta_{\mathfrak{s}}(\Omega^T_t)} \lesssim \| \mathcal{R}F - P(\tilde{F}^{<\beta}, \Pi)) \|_{\mathcal{C}^\beta_{\mathfrak{s}}(\Omega^T_t)} \\
&+ \| P(\tilde{F}^{<\beta}, \Pi) - a(D)P(\mathcal{I}(\tilde{F}^{<\beta}), \Pi) \|_{\mathcal{C}^\beta_{\mathfrak{s}}(\Omega^T_t)} \\
&+ \| a(D)(P(\mathcal{I}(\tilde{F}^{<\beta}), \Pi) - P(\overline{\mathcal{I}(F^{<\beta})}, \Pi)) \|_{\mathcal{C}^\beta_{\mathfrak{s}}(\Omega^T_t)} \\
&\lesssim (\| \tilde{F}^{<\beta} \|_{\mathcal{D}^\beta(\mathbb{R}^d;\mathcal{T})} \\
&+ t^{\frac{\eta-\beta}{\theta}} \| F \|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V})}) \lesssim t^{\frac{\eta-\beta}{\theta}} \| F \|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V})} \,,
\end{aligned}$$

where we used that by uniqueness of the reconstruction operator in Theorem 2.3.19 $\mathcal{R}\tilde{F}^{<\beta} = \mathcal{R}F$ in Ω^T_t. Part (b) is proved.

For Part (c) define once more $\tilde{F}^{<\beta-\theta} = \mathcal{E}_{\Omega^T_t} F^{<\beta-\theta}$. We have

$$\begin{aligned}
\| \mathcal{I}(\tilde{F}^{<\beta-\theta}) \|_{\mathcal{D}^\beta(\mathbb{R}^d;\mathcal{W}\backslash\overline{\mathcal{T}})} &\lesssim \| \mathcal{I}(F^{<\beta-\theta}) \|_{\mathcal{D}^\beta(\Omega^T_t;\mathcal{W}\backslash\overline{\mathcal{T}})} = \| \mathcal{I}^{<\beta}(F) \|_{\mathcal{D}^\beta(\Omega^T_t;\mathcal{W}\backslash\overline{\mathcal{T}})} \\
&\overset{\text{(a)}}{\lesssim} T^{\frac{\kappa}{\theta}} t^{\frac{\tilde{\eta}-\beta}{\theta}} \| F \|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V})} \leqslant T^{\frac{\kappa}{2\theta}} t^{\frac{\tilde{\eta}-\beta}{\theta}} \| F \|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V})} \,, \qquad (6.39)
\end{aligned}$$

where we used in the last step that we assumed $T \leqslant 1$. In the following $\mathscr{R}$ will denote a term that will change from line to line and satisfies $\|\mathscr{R}\|_{\mathcal{C}^{\beta-|\mu|_{\mathfrak{s}}}_{\mathfrak{s}}(\Omega^T_t)} \lesssim t^{\frac{\tilde{\eta}-\beta}{\theta}} T^{\frac{\kappa}{2\theta}} \|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V})}$. We can then, with Part (b), Proposition 5.3.13, (6.39) and the fact that $\Gamma^{\mathbf{1}}_{ux} \mathcal{I}(F^{>\beta-\theta}_x) = \Gamma^{\mathbf{1}}_{ux} \mathcal{I}^{>\beta}(F_x) = \mathcal{O}(\|x-u\|^\beta_{\mathfrak{s}})$, reshape (6.31) to

$$\begin{aligned}
\mathcal{P}^{X^\mu}_x F &= \lim_{N\to\infty} \frac{1}{\mu!} \int \mathrm{d}u \, \partial^\mu \Psi^{<N}_{x-u} \big((I\mathcal{R}F)_u - \Gamma^{\mathbf{1}}_{ux} \mathcal{I}(F_x) \big) = \frac{1}{\mu!} \lim_{N\to\infty} \int \mathrm{d}u \, \partial^\mu \Psi^{<N}_{x-u} \\
&\quad \times ((I\mathcal{R}F)_u - P(\overline{\mathcal{I}(F^{<\beta-\theta})}, \Pi)_u + P(\overline{\mathcal{I}(F^{<\beta-\theta})}, \Pi)_u - P(\mathcal{I}(\tilde{F}^{<\beta-\theta}, \Pi)_u \\
&\qquad + P(\mathcal{I}(\tilde{F}^{<\beta-\theta}, \Pi)_u - \Gamma^{\mathbf{1}}_{ux} \mathcal{I}(F^{<\beta-\theta}_x)) \\
&= \mathscr{R} + \lim_{N\to\infty} \frac{1}{\mu!} \int \mathrm{d}u \, \partial^\mu \Psi^{<N}_{x-u} (P(\mathcal{I}(\tilde{F}^{<\beta}, \Pi)_u - \Gamma^{\mathbf{1}}_{ux} \mathcal{I}(F^{<\beta-\theta}_x)) \qquad (6.40)
\end{aligned}$$

for $x \in \Omega_t^T$. To estimate the remaining term some kneading is needed. To this end we use that for any $K \geqslant 1$ and $x \in \Omega_t^T$ one has

$$\begin{aligned}
&\lim_{N\to\infty} \int \mathrm{d}u \left(\partial^\mu \Psi^{<N}_{x-u} \Gamma^{\mathbf{1}}_{ux} \mathcal{I}(F_x^{<\beta-\theta}) - \Psi^{<N}_{x-u} \int \mathrm{d}v\mathrm{d}w\, \Psi^{\leqslant N+K}_{u-v} \partial^\mu \Psi^{\leqslant N+K}_{u-w} \Gamma^{\mathbf{1}}_{wv} \mathcal{I}(F_v^{<\beta-\theta})\right) \\
&\overset{(*_1)}{=} \lim_{N\to\infty} \int \mathrm{d}u \Psi^{<N}_{x-u} \Big(\iint \mathrm{d}v\mathrm{d}w\, \Psi^{\leqslant N+K}_{u-v} \partial^\mu \Psi^{\leqslant N+K}_{u-w} \left(\Gamma^{\mathbf{1}}_{wx} \mathcal{I}(F_x^{<\beta-\theta}) - \Gamma^{\mathbf{1}}_{wv} \mathcal{I}(\tilde{F}_v^{<\beta-\theta})\right) \Big) \\
&\overset{(*_2)}{=} 0\,, \qquad (6.41)
\end{aligned}$$

For $(*_1)$ we used the spectral support of $\partial^\mu \Psi^{<N}_{x-u}$ to insert in the first term a convolution with $\Psi^{\leqslant N+K}_{u-w}$, integrated by parts and then smuggled in $\int \mathrm{d}v\, \Psi^{\leqslant N+K}_{u-v} = 1$. In $(*_2)$ we used that due to $|\mu|_{\mathfrak{s}} < \beta$ and $\mathcal{I}(\tilde{F}^{<\beta-\theta}) \in \mathcal{D}^\beta(\mathbb{R}^d; \mathcal{W}\backslash\overline{\mathcal{T}})$ (which is true by (6.39)) the first term of

$$\begin{aligned}
\Gamma^{\mathbf{1}}_{wx} \mathcal{I}(F_x^{<\beta-\theta}) - \Gamma^{\mathbf{1}}_{wv} \mathcal{I}(\tilde{F}_v^{<\beta-\theta}) &= \Gamma^{\mathbf{1}}_{wx} \left[\mathcal{I}(F_x^{<\beta-\theta}) - \Gamma^{\mathcal{W}\backslash\overline{\mathcal{T}}}_{xv} \mathcal{I}(\tilde{F}_v^{<\beta-\theta})\right] \\
&\quad - \Gamma^{\mathbf{1}}_{wx} \Gamma^{\overline{\mathcal{T}}}_{xv} \mathcal{I}(F_v^{<\beta-\theta}) \qquad (6.42)
\end{aligned}$$

vanishes in the limit, while the second is smooth in w and slightly Hölder continuous in v so that only

$$-\partial^\mu \Gamma^{\mathbf{1}}_{wx} \Gamma^{\overline{\mathcal{T}}}_{xx} \mathcal{I}(F_x^{\beta-\theta}) = -\mu!\, \Gamma^{X^\mu}_{xx} \mathcal{I}(F_x) = 0$$

remains in the limit $N \to \infty$.

Note further that since $\mathcal{I}(\tilde{F}^{<\beta-\theta})$ is function-like (compare the definition of the Whitney extension (5.69)) we have by Lemma 6.3.6 below $P(\mathcal{I}(\tilde{F}^{<\beta-\theta}), \Pi) = P(\mathcal{I}(\tilde{F}^{<\beta-\theta}), \Gamma^{\mathbf{1}})$. We can thus rewrite for some $K \geqslant 1$ by spectral support properties and Leibniz's rule

$$\begin{aligned}
\int \mathrm{d}u\, \partial^\mu \Psi^{<N}_{x-u} P(\mathcal{I}(\tilde{F}^{<\beta-\theta}, \Pi)_u = &\sum_{0\leqslant\nu\leqslant\mu} \int \mathrm{d}u\, \Psi^{<N}_{x-u} \binom{\mu}{\nu} \\
&\times \sum_{0<j\leqslant N+K} \iint \partial^\nu \Psi^{<j-1}_{u-v} \partial^{\mu-\nu} \Psi^{j}_{u-w} \Gamma^{\mathbf{1}}_{wv} \mathcal{I}(\tilde{F}_v^{<\beta-\theta}) \quad (6.43)
\end{aligned}$$

The terms with $\nu > 0$ converge for $N \to 0$ by Lemma 6.3.4 below to objects that can be swallowed in $\mathscr{R}$. Consequently, we are only left in (6.40), using (6.41) and the $\nu = 0$ term of (6.43), with

$$\mathcal{P}^{X^\mu}_x F = \mathscr{R} - \lim_{N\to\infty} \frac{1}{\mu!} \int \mathrm{d}u \Psi^{<N}_{x-u} \sum_{-1\leqslant i\leqslant N+K} \iint \mathrm{d}v\mathrm{d}w\, \Psi^{i}_{u-v} \partial^\mu \Psi^{\leqslant i+1}_{u-w} \Gamma^{\mathbf{1}}_{wv} \mathcal{I}(\tilde{F}_v^{<\beta-\theta})\,. \tag{6.44}$$

But on the other hand we have by definition of the paraproduct

$$P(\mathcal{I}(\tilde{F}^{<\beta-\theta}),\Gamma^{X^\mu})_x + \sum_{i\geqslant -1}\iint \mathrm{d}v\mathrm{d}w\Psi^i_{x-v}\Psi^{\leqslant i+1}_{x-w}\Gamma^{X^\mu}_{wv}\mathcal{I}(\tilde{F}^{<\beta-\theta}_v) = \Gamma^{X^\mu}_{xx}\mathcal{I}(F^{<\beta-\theta}_x) = 0$$

so that with K as above

$$\begin{aligned} P(\mathcal{I}(\tilde{F}^{<\beta-\theta}),\Gamma^{X^\mu})_x &= \sum_{i\geqslant -1}\iint \mathrm{d}v\mathrm{d}w\Psi^i_{x-v}\Psi^{\leqslant i+1}_{x-w}\Gamma^{X^\mu}_{wv}\mathcal{I}(\tilde{F}^{<\beta-\theta}_v) \\ &= \lim_{N\to\infty}\int \mathrm{d}u\Psi^{<N}_{x-u}\sum_{-1\leqslant i\leqslant N+K}\iint \mathrm{d}v\mathrm{d}w\,\Psi^i_{u-v}\partial^\mu\Psi^{\leqslant i+1}_{u-w}\Gamma^{X^\mu}_{wv}\mathcal{I}(\tilde{F}^{<\beta-\theta}_v)\,. \end{aligned} \tag{6.45}$$

Subtracting (6.45) from (6.44) we obtain

$$\begin{aligned} \mathcal{P}^{X^\mu}_x F - P(\mathcal{I}(\tilde{F}^{<\beta-\theta}),\Gamma^{X^\mu})_x &= \mathscr{R} - \frac{1}{\mu!}\lim_{N\to\infty}\int \mathrm{d}u\Psi^{<N}_{x-u}\sum_{-1\leqslant i\leqslant N+K}\iint \mathrm{d}v\mathrm{d}w\,\Psi^i_{u-v} \\ &\quad\times\left(\partial^\mu\Psi^{\leqslant i+1}_{u-w}\Gamma^{\mathbf{1}}_{wv}\mathcal{I}(\tilde{F}^{<\beta-\theta}_v) - \mu!\Psi^{\leqslant i+1}_{u-w}\Gamma^{X^\mu}_{wv}\mathcal{I}(\tilde{F}^{<\beta-\theta}_v)\right). \end{aligned} \tag{6.46}$$

For $\mu = 0$ the right hand side vanishes so that we assume from now on $\mu \neq 0$. All the terms in the sum of the right hand are spectrally supported in a box of size $2^{i\mathfrak{s}}\mathcal{B}$, so that by Lemma 2.1.19 it remains to find a bound on

$$\iint \mathrm{d}v\mathrm{d}w\,\Psi^i_{u-v}\left(\partial^\mu\Psi^{\leqslant i+1}_{u-w}\Gamma^{\mathbf{1}}_{wv}\mathcal{I}(\tilde{F}^{<\beta-\theta}_v) - \Psi^{\leqslant i+1}_{u-w}\Gamma^{X^\mu}_{wv}\mathcal{I}(\tilde{F}^{<\beta-\theta}_v)\right).$$

For $i = -1$ this is easy. For $i \geqslant 0$ we achieve this goal by reshaping this expression to

$$\begin{aligned} &\iint \mathrm{d}v\mathrm{d}w\,\Psi^i_{u-v}\Big[\partial^\mu\Psi^{\leqslant i+1}_{u-w}\Gamma^{\mathbf{1}}_{wu}(\Gamma^{\mathcal{W}\setminus\overline{\mathcal{T}}}_{uv}\mathcal{I}(\tilde{F}^{<\beta-\theta}_v) - \mathcal{I}(\tilde{F}^{<\beta-\theta}_u)) \\ &\qquad\qquad - \mu!\Psi^{\leqslant i+1}_{u-w}\Gamma^{X^\mu}_{wu}(\Gamma^{\mathcal{W}\setminus\overline{\mathcal{T}}}_{uv}\mathcal{I}(\tilde{F}^{<\beta-\theta}_v) - \mathcal{I}(\tilde{F}^{<\beta-\theta}_u))\Big] \\ &+ \iint \mathrm{d}v\mathrm{d}w\,\Psi^i_{u-v}\Big[\partial^\mu\Psi^{\leqslant i+1}_{u-w}\Gamma^{\mathbf{1}}_{wu}\Gamma^{\overline{\mathcal{T}}}_{uv}\mathcal{I}(\tilde{F}^{<\beta-\theta}_v) - \mu!\Psi^{\leqslant i+1}_{u-w}\Gamma^{X^\mu}_{wu}\Gamma^{\overline{\mathcal{T}}}_{uv}\mathcal{I}(\tilde{F}^{<\beta-\theta}_v)\Big], \end{aligned}$$

where we used $\int \mathrm{d}v\Psi^i_{u-v} = 0$ to introduce twice a term $\mathcal{I}(\tilde{F}^{<\beta-\theta}_u)$ (independent of v!) and where we split $\Gamma^{\mathbf{1}}_{wv}\mathcal{I}(\tilde{F}^{<\beta-\theta}_v) = \Gamma^{\mathbf{1}}_{wu}\Gamma^{\mathcal{W}\setminus\overline{\mathcal{T}}}_{uv}\mathcal{I}(\tilde{F}^{<\beta-\theta}_v) + \Gamma^{\mathbf{1}}_{wu}\Gamma^{\overline{\mathcal{T}}}_{uv}\mathcal{I}(\tilde{F}^{<\beta-\theta}_v)$ and similar for $\Gamma^{X^\mu}_{wv}\mathcal{I}(\tilde{F}^{<\beta-\theta}_v)$. The first term can once more be bounded using (6.39) by $2^{-i(\beta-|\mu|_\mathfrak{s})}\,\|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{T})}\,t^{\frac{\tilde{\eta}-\beta}{\theta}}T^{\frac{\kappa}{2\theta}}$, while the second term vanishes due to Lemma 2.1.15. Consequently

$$\mathcal{P}^{X^\mu}F - P(\mathcal{I}(\tilde{F}^{<\beta-\theta}),\Gamma^{X^\mu}) = \mathscr{R}$$

and the first bound of (c) is proved. The second bound is then an easy consequence of the observation

$$\begin{aligned}\|\mathcal{P}^{X^\mu}F\|_{C_b(\Omega_t^T;\mathbb{R})} &\lesssim \big\|\mathcal{P}^{X^\mu}(F) - P\big(\mathcal{I}(\tilde{F}^{<\beta-\theta}), \Gamma^{X^\mu}\big)\big\|_{\mathcal{C}_{\mathfrak{s}}^{\beta-|\mu|_{\mathfrak{s}}}(\Omega_t^T)} \\ &\quad + \|P(\mathcal{I}(\tilde{F}^{<\beta-\theta}), \Gamma^{X^\mu})\|_{C_b(\Omega_t^T;\mathbb{R})} \\ &\lesssim T^{\frac{\kappa}{2\theta}} t^{\frac{\tilde{\eta}-\beta}{\theta}} \|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V})} + \|\mathcal{I}(\tilde{F}^{<\beta-\theta})\|_{\mathcal{D}^\beta(\mathbb{R}^d;\mathcal{W}\setminus\overline{\mathcal{T}})} \\ &\overset{(6.39)}{\lesssim} T^{\frac{\kappa}{2\theta}} t^{\frac{\tilde{\eta}-\beta}{\theta}} \|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V})}\,,\end{aligned}$$

where we used that $P(\mathcal{I}(\tilde{F}^{<\beta-\theta}), \Gamma^{X^\mu}) \in \mathcal{C}_{\mathfrak{s}}^\kappa(\mathbb{R}^d)$ for $0 < \kappa < \min\{\alpha : |\mu|_{\mathfrak{s}} < \alpha\} - |\mu|_{\mathfrak{s}}$. Using the representations (6.46) and (6.45) we see in particular that the sequence in (6.31) is convergent.

For Part (d) we have to show that $\mathcal{K}F$ satisfies (5.13) for $\alpha' \in A_{\mathcal{W}} \cap |\mathbb{N}^d|_{\mathfrak{s}}$ and $\overline{\gamma}$. Fix $l \in \mathbb{N}^d_{<\overline{\gamma}}$ such that $|l|_{\mathfrak{s}} = \alpha'$, some $k \in \mathbb{N}^d$ with $0 \leqslant |k|_{\mathfrak{s}} < \overline{\gamma} - \alpha' = \overline{\gamma} - |l|_s$ and $t \in (0,T)$, $x \in \Omega_t^T$. We have to show

$$\begin{aligned}0 &\overset{!}{=} \lim_{N\to\infty} \int \mathrm{d}u\, \partial^k \Psi^{<N}_{x-u}\, (\mathcal{P}_u^{X^l} F - \Gamma_{ux}^{X^l} \mathcal{K}_x F) \\ &= \lim_{N\to\infty} \int \mathrm{d}u\, \partial^k \Psi^{<N}_{x-u}\, (\mathcal{P}_u^{X^l} F - \Gamma_{ux}^{X^l} \mathcal{I}(F_x)) - \frac{(l+k)!}{l!} \mathcal{P}_x^{X^{k+l}} F\end{aligned}$$

In other words we have to show

$$\lim_{N\to\infty} \int \mathrm{d}u\, \partial^k \Psi^{<N}_{x-u}\, (\mathcal{P}_u^{X^l} F - \Gamma_{ux}^{X^l} \mathcal{I}(F_x)) \overset{!}{=} \frac{(l+k)!}{l!} \mathcal{P}_x^{X^{k+l}}(F) \tag{6.47}$$

Similar as in Part (c) we can drop all the components F^α with $\alpha > \overline{\gamma} - \theta$, so that we can assume without loss of generality $F = F^{<\overline{\gamma}-\theta}$. To show (6.47) we use that by definition of $\mathcal{P}F$ we can rewrite the left hand side as

$$\begin{aligned}&\lim_{N\to\infty} \int \mathrm{d}u\, \partial^k \Psi^{<N}_{x-u} \Big(\frac{1}{l!} \lim_{\tilde{N}\to\infty} \int \mathrm{d}v\, \partial^l \Psi^{<\tilde{N}}_{v-u} (\mathcal{P}_v^{\mathbf{1}}(F) - \Gamma_{vu}^{\mathbf{1}} \mathcal{I}(F_u)) - \Gamma_{ux}^{X^l} \mathcal{I}(F_x)\Big) \\ &= \lim_{N\to\infty} \int \mathrm{d}u\, \partial^k \Psi^{<N}_{x-u}\, \mathbf{1}_{\Omega_t^T}(u) \Big(\frac{1}{l!} \lim_{\tilde{N}\to\infty} \int \mathrm{d}v\, \partial^l \Psi^{<\tilde{N}}_{v-u} (\mathcal{P}_v^{\mathbf{1}}(F) - \Gamma_{vu}^{\mathbf{1}} \mathcal{I}(\tilde{F}_u)) - \Gamma_{ux}^{X^l} \mathcal{I}(F_x)\Big)\end{aligned} \tag{6.48}$$

where $\tilde{F} = \mathcal{E}_{\Omega_t^T} F \in \mathcal{D}^{\overline{\gamma}-\theta}(\mathbb{R}^d; \mathcal{T})$ and where we used $\Psi^{<N}$ is the scaled version of a Schwartz function to cut-off the integral for $u \notin \Omega_t^T$ in the limit $N \to \infty$. The key observation is now that the inner sequence has convergence rate $2^{-\tilde{N}(\overline{\gamma}-|l|_{\mathfrak{s}})}$. Indeed, consider the remainder

$$\sum_{i \geqslant \tilde{N}} \frac{1}{l!} \int \mathrm{d}v\, \partial^k \Psi^i_{u-v} \Big(\mathcal{P}_v^{\mathbf{1}} F - P(\mathcal{I}(\tilde{F}), \Pi)_v + P(\mathcal{I}(\tilde{F}, \Gamma^{\mathbf{1}})_v - \Gamma_{vu}^{\mathbf{1}} \mathcal{I}(\tilde{F}_u)\Big) \tag{6.49}$$

By Part (b), Proposition 5.3.13 and the fact that $\mathcal{I}(\tilde{F}) \in \mathcal{D}^{\overline{\gamma}}(\mathbb{R}^d; \mathcal{W}\backslash\overline{\mathcal{T}})$ (by (6.36)) we have $v \mapsto f(v) := \mathcal{P}^{\mathbf{1}}_v F - P(\mathcal{I}(\tilde{F}), \Pi)_v \in \mathcal{C}^{\overline{\gamma}}_{\mathfrak{s}}(\Omega^T_t)$. Exploiting that Ψ^i_{u-v} becomes focussed (around $u \in \Omega^T_t$) for $i \to \infty$ we can replace this function in the integral by an arbitrary at most polynomial growing extension $v \mapsto \tilde{f}(v) \in \mathcal{C}^{\overline{\gamma}}_{\mathfrak{s}}(\mathbb{R}^d)$ with $\tilde{f}|_{\Omega^T_t} = f$, for example by lifting the object in the polynomial regularity structure and then applying the Whitney extension $\mathcal{E}_{\Omega^T_t}$ from Theorem 5.3.16. This shows, by Definition of $\mathcal{C}^{\overline{\gamma}}_{\mathfrak{s}}(\mathbb{R}^d)$ and Lemma 2.1.33, that the first part of (6.49) does indeed vanish like $2^{-\tilde{N}(\overline{\gamma}-|k|_{\mathfrak{s}})}$. For the second part this is also true as we can reshape it as

$$\sum_{i \geqslant \tilde{N}} \frac{1}{l!} \int \mathrm{d}v\, \partial^k \Psi^i_{u-v} \sum_{j:\, j \sim i} \iint \mathrm{d}z_1 \mathrm{d}z_2 \Psi^{<j-1}_{v-z_1} \Psi^j_{v-z_2}\, \Gamma^{\mathbf{1}}_{z_2 z_1} (\mathcal{I}(\tilde{F}_{z_1}) - \Gamma_{z_1 u} \mathcal{I}(\tilde{F}_u))$$

which also, after omitting as usual polynomial contribution, and applying once more $\mathcal{I}(\tilde{F}) \in \mathcal{D}^{\overline{\gamma}}(\mathbb{R}^d; \mathcal{W}\backslash\overline{\mathcal{T}})$ vanishes with rate $2^{-\tilde{N}(\overline{\gamma}-|k|_{\mathfrak{s}})}$. With this knowledge the limit in (6.48) must equal

$$\begin{aligned}
&\lim_{N\to\infty} \int \mathrm{d}u\, \partial^k \Psi^{<N}_{x-u}\, \mathbf{1}_{\Omega^T_{t/2}}(u) \frac{1}{l!} \Big(\int \mathrm{d}v\, \partial^l \Psi^{<N+2}_{v-u} (\mathcal{P}^{\mathbf{1}}_v F - \Gamma^{\mathbf{1}}_{vu} \mathcal{I}(\tilde{F}_u)) - \Gamma^{X^l}_{ux} \mathcal{I}(F_x) \Big) \\
&\quad = \lim_{N\to\infty} \int \mathrm{d}u\, \partial^k \Psi^{<N}_{x-u} \frac{1}{l!} \Big(\int \mathrm{d}v\, \partial^l \Psi^{<N+2}_{v-u} ((\mathcal{P}^{\mathbf{1}}_v F - \Gamma^{\mathbf{1}}_{vu} \mathcal{I}(\tilde{F}_u)) - \Gamma^{X^l}_{ux} \mathcal{I}(F_x) \Big)
\end{aligned} \tag{6.50}$$

where we used once more $\Psi^{\leqslant N} \in \mathcal{S}(\mathbb{R}^d)$ to erase the indicator function. Inserting $-\Gamma^{\mathbf{1}}_{vx} \mathcal{I}(\tilde{F}_x) + \Gamma^{\mathbf{1}}_{vx} \mathcal{I}(\tilde{F}_x)$ and applying Lemma 2.1.15 transforms (6.50) to

$$\begin{aligned}
&\lim_{N\to\infty} \int \mathrm{d}u\, \partial^k \Psi^{<N}_{x-u} \Big(\frac{1}{l!} \int \mathrm{d}v\, \partial^l \Psi^{<N+2}_{v-u} \\
&\times \big(\mathcal{P}^{\mathbf{1}}_v F - \Gamma^{\mathbf{1}}_{vx} \mathcal{I}(F_x) + \Gamma^{\mathbf{1}}_{vu} [\Gamma^{\mathcal{W}\backslash\overline{\mathcal{T}}}_{ux} \mathcal{I}(F_x) + \Gamma^{\overline{\mathcal{T}}}_{ux} \mathcal{I}(F_x) - \mathcal{I}(\tilde{F}_u)] \big) - \Gamma^{X^l}_{ux} \mathcal{I}(F_x) \Big) \\
&\overset{\text{Lem. 2.1.15}}{=} \lim_{N\to\infty} \int \mathrm{d}u\, \partial^k \Psi^{<N}_{x-u} \\
&\Big(\frac{1}{l!} \int \mathrm{d}v\, \partial^l \Psi^{<N+2}_{v-u} (\mathcal{P}^{\mathbf{1}}_v F - \Gamma^{\mathbf{1}}_{vx} \mathcal{I}(F_x) + \Gamma^{\mathbf{1}}_{vu} [\Gamma^{\mathcal{W}\backslash\overline{\mathcal{T}}}_{ux} \mathcal{I}(F_x) - \mathcal{I}(\tilde{F}_u)] \Big)\,,
\end{aligned}$$

where we used that by Lemma 2.1.15 $\frac{1}{l!} \int \mathrm{d}v\, \partial^l \Psi^{<N+2}_{v-u}\, \Gamma^{\mathbf{1}}_{vu} \Gamma^{\overline{\mathcal{T}}}_{ux} \mathcal{I}(F_x) = \Gamma^{X^l}_{ux} \mathcal{I}(F_x)$. The contribution of $\Gamma^{\mathbf{1}}_{vu}(\Gamma^{\mathcal{W}\backslash\overline{\mathcal{T}}}_{ux} \mathcal{I}(F_x) - \mathcal{I}(\tilde{F}_u))$ vanishes in the limit, so that we are left with

$$\begin{aligned}
&\frac{1}{l!} \lim_{N\to\infty} \iint \mathrm{d}u\, \mathrm{d}v\, \partial^k \Psi^{<N}_{x-u}, \partial^l \Psi^{<N+2}_{v-u} (\mathcal{P}^{\mathbf{1}}_v F - \Gamma^{\mathbf{1}}_{vx} \mathcal{I}(F_x)) \\
&= \frac{1}{l!} \lim_{N\to\infty} \int \mathrm{d}u\, \partial^{k+l} \Psi^{<N}_{x-v} (\mathcal{P}^{\mathbf{1}}_v(F) - \Gamma^{\mathbf{1}}_{vx} \mathcal{I}(F_x)) = \frac{(l+k)!}{l!} \mathcal{P}^{X^{k+l}}_x(F)\,,
\end{aligned}$$

where we used that by spectral support properties and symmetry of the functions $(\Psi^j)_{j\geqslant -1}$ we have $\Psi^{<N} * \Psi_{-\cdot}^{<N+2} = \Psi^{<N} * \Psi^{<N+2} = \Psi^{<N}$ which finishes the proof for (6.34). The distance estimate (6.35) follows by literally the same arguments as in Parts (a)-(c) by using the corresponding distance estimates from the applied lemmas, propositions and theorems instead. □

6.3 Technical Results

Lemma 6.3.1. *The terms $R^{k,\gamma}_{x;h}$ in Lemma 2.1.20 can be written as*

$$R^{k,\gamma}_{x;h} f = r^{\gamma,k}_{x;h}(f) \cdot h^{k-e_{\mathfrak{m}(k)}} \tag{6.51}$$

with $|r^{\gamma,k}_{x;h}(f)| \lesssim \sup_{\zeta\in[0,h]} |\partial^{k-e_{\mathfrak{m}(k)}} f(x+v^k_\zeta(h) - \partial^{k-e_{\mathfrak{m}(k)}} f(x+v^k_0(h))|$. In particular if $x, x+h \in \Omega$ for some convex set Ω we have for $k \in \mathbb{N}^d_{>\gamma}$

$$|R^{k,\gamma}_{x;h} f| \lesssim \|\partial^{k-e_{\mathfrak{m}(k)}} f\|^{1-\rho_{\gamma,k}}_{C_b(\Omega;\mathbb{R})} \|\partial^k f\|^{\rho_{\gamma,k}}_{C_b(\Omega;\mathbb{R})} \|h\|^\gamma_{\mathfrak{s}} \tag{6.52}$$

with $\rho_{\gamma,k} = \frac{\gamma - |k-e_{\mathfrak{m}(k)}|_{\mathfrak{s}}}{\mathfrak{s}_{\mathfrak{m}(k)}}$.

Proof. We will actually show that $r^{\gamma,k}_{x;h}(f)$ are of the form

$$r^{\gamma,k}_{x;h}(f) = \begin{cases} \frac{1}{(k-2e_{\mathfrak{m}(k)})!} \int_0^1 \mathrm{d}\zeta\,(1-\zeta)^{k_{\mathfrak{m}(k)}-2} [\partial^{k-e_{\mathfrak{m}(k)}} f(x+v^k_\zeta(h) - \partial^{k-e_{\mathfrak{m}(k)}} f(x+v^k_0(h)] & \text{if } k_{\mathfrak{m}(k)} \geqslant 2 \\ \frac{1}{(k-e_{\mathfrak{m}(k)})!} [\partial^{k-e_{\mathfrak{m}(k)}} f(x+v^k_1(h)) - \partial^{k-e_{\mathfrak{m}(k)}} f(x+v^k_0(h))] & \text{if } k_{\mathfrak{m}(k)} = 1 \end{cases}$$

from with the claimed bound for $r^{\gamma,k}_{x;h}(f)$ obviously follows. For $k_{\mathfrak{m}(k)} = 1$ this is just the fundamental theorem of calculus. For $k_{\mathfrak{m}(k)} \geqslant 2$ we reshape the formula given above by rewriting it as

$$\frac{1}{(k-2e_{\mathfrak{m}(k)})!} \int_0^1 \mathrm{d}\zeta\,(1-\zeta)^{k_{\mathfrak{m}(k)}-2} \int_0^\zeta \mathrm{d}\zeta'\, \partial^k f(x+v^k_{\zeta'}(h))$$

which equals $R^{k,\gamma}_{x;h} f$ after an integration by parts in the outer integral. To show the interpolation result (6.52) wse that we know from Lemma 2.1.20 that $|R^{k,\gamma}_{x;h} f| \lesssim \|\partial^k f\|_{C_b(\Omega)} \|h\|^{|k|_{\mathfrak{s}}}_{\mathfrak{s}}$, so that

$$\begin{aligned} |R^{k,\gamma}_{x;h} f| = |R^{k,\gamma}_{x;h} f|^{1-\rho_{\gamma,k}} |R^{k,\gamma}_{x;h} f|^{\rho_{\gamma,k}} &\lesssim \|\partial^{k-e_{\mathfrak{m}(k)}} f\|^{1-\rho_{\gamma,k}}_{C_b(\Omega;\mathbb{R})} \|h\|^{(1-\rho_{\gamma,k})|k-e_{\mathfrak{m}(k)}|_{\mathfrak{s}}}_{\mathfrak{s}} \\ &\quad \times \|\partial^k f\|^{\rho_{\gamma,k}}_{C_b(\Omega;\mathbb{R})} \|h\|^{\rho_{\gamma,k}|k|_{\mathfrak{s}}}_{\mathfrak{s}}, \end{aligned}$$

which equals (6.52) due to $(1-\rho)|k-e_{\mathfrak{m}(k)}|_{\mathfrak{s}} + \rho|k|_{\mathfrak{s}} = (1-\rho)(|k|_{\mathfrak{s}} - \mathfrak{s}_{\mathfrak{m}(k)}) + \rho|k|_{\mathfrak{s}} = |k|_{\mathfrak{s}} - \mathfrak{s}_{\mathfrak{m}(k)} \frac{\gamma - |k|_{\mathfrak{s}}}{\mathfrak{s}_{\mathfrak{m}(k)}} = \gamma$. □

Lemma 6.3.2. *Let $\mu \in \mathbb{R}\backslash|\mathbb{N}^d|_{\mathfrak{s}}$, $\beta \geqslant 0$ such that $\beta + \mu \notin |\mathbb{N}^d|_{\mathfrak{s}}$, we then have for $f \in \mathcal{C}^\mu(\mathbb{R}^{d-1})$ for Ω_t^T as in Section 5.3 and $\theta \geqslant 1$ the bound*

$$\|x = (x_1, x') \mapsto (e^{x_1 p(D')} f)(x')\|_{\mathcal{C}^{\mu+\beta}_{(\theta,1,\dots,1)}(\Omega_t^T;\mathbb{R})} \lesssim t^{-\frac{\beta}{\theta}} \|f\|_{\mathcal{C}^{\mu}_{(1,\dots,1)}(\mathbb{R}^{d-1};\mathbb{R})} .$$

Proof. It follows from (a modification of) [BCD11, Lemma 2.4] that for $t > 0$

$$\|e^{tp(D')} f\|_{\mathcal{C}^{\mu+\beta}_{(1,\dots,1)}(\mathbb{R}^{d-1};\mathbb{R})} \lesssim t^{-\frac{\beta}{\theta}} \|f\|_{\mathcal{C}^{\mu}_{(1,\dots,1)}(\mathbb{R}^{d-1};\mathbb{R})} .$$

This implies in particular for $\nu > 0$ and $g \in \mathcal{C}^{-\nu}(\mathbb{R}^{d-1})$, by choosing $j' \geqslant 0$ such that $2^{-j'-1} < t^{1/\theta} \leqslant j'$ and an arbitrary $\delta > 0$,

$$\begin{aligned}
\|e^{tp(D')} g\|_{C_b(\mathbb{R}^{d-1};\mathbb{R})} &\leqslant \sum_{j\leqslant j'} \|e^{tp(D')} \varDelta_j g\|_{\mathcal{C}^{-\nu}_{(1,\dots,1)}(\mathbb{R}^{d-1};\mathbb{R})} + \sum_{j>j'} \|e^{tp(D')} \varDelta_j g\|_{\mathcal{C}^{\delta}_{(1,\dots,1)}(\mathbb{R}^{d-1};\mathbb{R})} \\
&\lesssim \Big(\sum_{j\leqslant j'} 2^{j\nu} + t^{\frac{-\nu-\delta}{\theta}} \sum_{j>j'} 2^{-j\delta} \Big) \|g\|_{\mathcal{C}^{-\nu}_{(1,\dots,1)}(\mathbb{R}^{d-1};\mathbb{R})} \lesssim t^{-\frac{\nu}{\delta}} \|g\|_{\mathcal{C}^{-\nu}_{(1,\dots,1)}(\mathbb{R}^{d-1};\mathbb{R})} .
\end{aligned} \tag{6.53}$$

Let us write $Pf(x_1, x') = e^{x_1 p(D')} f$ and note for any $k = (k_1, k') \in \mathbb{N} \times \mathbb{N}^{d-1}$ the identity $\partial^k Pf = P(p(D')^{k_1} \partial^{k'} f)$, which implies together with (6.53) that for any $k \in \mathbb{N}^d$ with $|k|_{\mathfrak{s}} > \mu$

$$\|\partial^k Pf\|_{C_b(\Omega_t^T;\mathbb{R})} \lesssim t^{\frac{\mu-|k|_{\mathfrak{s}}}{\theta}} \|f\|_{\mathcal{C}^{\mu}_{(1,\dots 1)}(\mathbb{R}^{d-1};\mathbb{R})} . \tag{6.54}$$

For the claim, we have to study for $x, x+h \in \Omega_t^T$ with $\|h\|_{\mathfrak{s}} \leqslant 1$

$$R^{\mu+\beta}_{x;h} Pf = \sum_{k\in\mathbb{N}^d_{>\mu+\beta}} R^{\mu+\beta,k}_{x;h} Pf = \sum_{\substack{k\,\in\,\mathbb{N}^d_{>\mu+\beta} \\ \mu\,<\,|k-e_{\mathfrak{m}(k)}|}} R^{\mu+\beta,k}_{x;h} Pf + \sum_{\substack{k\,\in\,\mathbb{N}^d_{>\mu+\beta} \\ \mu\,>\,|k-e_{\mathfrak{m}(k)}|}} R^{\mu+\beta,k}_{x;h} Pf \tag{6.55}$$

The first term can be estimated by first applying Lemma 6.3.1 and then (6.54)

$$\begin{aligned}
&\sum_{\substack{k\,\in\,\mathbb{N}^d_{>\mu+\beta} \\ \mu\,<\,|k-e_{\mathfrak{m}(k)}|}} \|\partial^{k-e_{\mathfrak{m}(k)}} Pf\|^{1-\rho_{\mu+\beta,k}}_{C_b(\Omega;\mathbb{R})} \|\partial^k Pf\|^{\rho_{\mu+\beta,k}}_{C_b(\Omega;\mathbb{R})} \|h\|^{\mu+\beta}_{\mathfrak{s}} \\
&\lesssim \sum_{\substack{k\,\in\,\mathbb{N}^d_{>\mu+\beta} \\ \mu\,<\,|k-e_{\mathfrak{m}(k)}|}} t^{(1-\rho_{\mu+\beta,k})\frac{\mu-|k-e_{\mathfrak{m}(k)}|_{\mathfrak{s}}}{\theta} + \rho_{\mu+\beta,k}\frac{\mu-|k|_{\mathfrak{s}}}{\theta}} \|h\|^{\mu+\beta}_{\mathfrak{s}} \|f\|_{\mathcal{C}^{\mu}_{(1,\dots,1)}(\mathbb{R}^{d-1};\mathbb{R})} \\
&\lesssim t^{-\frac{\beta}{\theta}} \|h\|^{\mu+\beta}_{\mathfrak{s}} \|f\|_{\mathcal{C}^{\mu}_{(1,\dots,1)}(\mathbb{R}^{d-1};\mathbb{R})} ,
\end{aligned}$$

where we used in the last step that $(1-\rho_{\mu+\beta,k})(\mu-|k-e_{\mathfrak{m}(k)}|_{\mathfrak{s}}) + \rho_{\mu+\beta,k}(\mu-|k|_{\mathfrak{s}}) = \mu - |k|_{\mathfrak{s}} + \mathfrak{s}_{\mathfrak{m}(k)}(1-\rho_{\mu+\beta,k}) = \beta$. To bound the second term of (6.55) we proceed in the same way but use that in this term $\|\partial^{k-e_{\mathfrak{m}(k)}} Pf\|_{C_b(\Omega;\mathbb{R})}^{1-\rho_{\mu+\beta,k}} \lesssim \|f\|_{\mathcal{C}^{\mu}_{(1,\dots,1)}(\mathbb{R}^{d-1};\mathbb{R})}^{1-\rho_{\mu+\beta,k}}$, which gives the bound

$$
\begin{aligned}
&\sum_{\substack{k\,\in\,\mathbb{N}^d_{>\mu+\beta}\\ \mu>|k-e_{\mathfrak{m}(k)}|}} \|f\|^{1-\rho_{\mu+\beta,k}}_{\mathcal{C}^{\mu}_{(1,\dots,1)}(\mathbb{R}^{d-1};\mathbb{R})} \|\partial^k Pf\|^{\rho_{\mu+\beta,k}}_{C_b(\Omega;\mathbb{R})} \|h\|^{\mu+\beta}_{\mathfrak{s}} \\
\lesssim \quad &\sum_{\substack{k\,\in\,\mathbb{N}^d_{>\mu+\beta}\\ \mu>|k-e_{\mathfrak{m}(k)}|}} t^{\rho_{\mu+\beta,k}\frac{\mu-|k|_{\mathfrak{s}}}{\theta}} \|h\|^{\mu+\beta}_{\mathfrak{s}} \|f\|_{\mathcal{C}^{\mu}_{(1,\dots,1)}(\mathbb{R}^{d-1};\mathbb{R})} \,,
\end{aligned}
$$

which can once more be bounded by $t^{-\frac{\beta}{\theta}} \|h\|^{\mu+\beta}_{\mathfrak{s}} \|f\|_{\mathcal{C}^{\mu}_{(1,\dots,1)}(\mathbb{R}^{d-1};\mathbb{R})}$ by using $\rho_{\mu+\beta,k}(\mu-|k|_{\mathfrak{s}}) = \frac{\mu-|k-e_{\mathfrak{m}(k)}|_{\mathfrak{s}}+\beta}{\mathfrak{s}_{\mathfrak{m}(k)}}(\mu-|k-e_{\mathfrak{m}(k)}|_{\mathfrak{s}}-\mathfrak{s}_{\mathfrak{m}(k)}) \geqslant -\beta$. The bounds on $\|\partial^k Pf\|_{C_b(\Omega_t^T}$ follow by similar but easier arguments. □

Lemma 6.3.3. *With the parameters from Section 6.2 we have for $\overline{\eta}' < \overline{\eta}$*

$$
\sup_{t\in(0,T)} \sup_{\overline{\beta}\in[\overline{\eta},\overline{\gamma}]\setminus A_{\mathbb{N}^d}} t^{\frac{\overline{\beta}-\overline{\eta}'}{\theta}} \|I_0 f\|_{\mathcal{C}^{\overline{\beta}}_{\mathfrak{s}}(\Omega_t^T)} \lesssim \sup_{t\in(0,T]} \sup_{\beta\in([\eta,\gamma]\setminus A_{\mathbb{N}^d})\cap(0,\gamma]} t^{\frac{\beta-\eta}{\theta}} \|f\|_{\mathcal{C}^{\beta}_{\mathfrak{s}}(\Omega_t^T)} \tag{6.56}
$$

Proof. We actually show a slightly stronger statement

$$
\sup_{t\in(0,T)} \sup_{\overline{\beta}\in[\eta+\theta,\gamma+\theta-\varepsilon]\setminus A_{\mathbb{N}^d}} t^{\frac{\overline{\beta}-(\eta+\theta-\varepsilon)}{\theta}} \|I_0 f\|_{\mathcal{C}^{\overline{\beta}}_{\mathfrak{s}}(\Omega_t^T)} \lesssim \sup_{t\in(0,T]} \sup_{\beta\in([\eta,\gamma]\setminus A_{\mathbb{N}^d})\cap(0,\gamma]} t^{\frac{\beta-\eta}{\theta}} \|f\|_{\mathcal{C}^{\beta}_{\mathfrak{s}}(\Omega_t^T)} \tag{6.57}
$$

for small $\varepsilon > 0$. The fact that the left hand side of (6.57) bounds the right hand side of (6.56) is essentially the same argument as in Lemma 5.3.8. We we will consider only $\varepsilon < \operatorname{dist}(\{\eta,\gamma,\eta+\theta,\gamma+\theta\}, A_{\mathbb{N}^d}) \wedge 1$ so that in particular $\gamma\pm\varepsilon$, $\eta\pm\varepsilon$, $\gamma+\theta\pm\varepsilon$, $\eta+\theta\pm\varepsilon \notin A_{\mathbb{N}^d}$. If $\eta < 0$ we further choose $\varepsilon > 0$ small enough such that $\eta-\varepsilon > -\theta$. We assume without loss of generality that the right hand side of (6.57) is bounded by 1. We can then conclude by interpolation if we can show for $\overline{\beta} \in \{\eta+\theta, \gamma+\theta-\varepsilon\}$ and $t \in (0,T)$ that

$$
\|I_0 f\|_{\mathcal{C}^{\overline{\beta}}_{\mathfrak{s}}(\Omega_t^T)} \lesssim t^{\frac{\eta+\theta-\varepsilon-\overline{\beta}}{\theta}} ,
$$

which by the Defintion of $\mathcal{C}^{\overline{\beta}}_{\mathfrak{s}}(\Omega_t^T)$ and Lemma 6.3.1 can be reformulated as the task to bound for x, $y \in \Omega_t^T$ and $k \in \mathbb{N}^d_{>\beta}$

$$
|\partial^{\overline{k}}(I_0 f)_y - \partial^{\overline{k}}(I_0 f)_x| \lesssim t^{\frac{\eta+\theta-\varepsilon-\overline{\beta}}{\theta}} \|y-x\|_{\mathfrak{s}}^{\overline{\beta}-|\overline{k}|_{\mathfrak{s}}} \tag{6.58}
$$

where we wrote $\overline{k} = k - e_{\mathfrak{m}(k)}$ (actually we also have bound $\|\partial^l I_0 f\|_{C_b(\Omega_t^T}$ for $l \in \mathbb{N}^d_{<\beta}$, but this is an easier version of the arguments used below). By definition we have $\partial_t f = I_0 p(D')f + f$ and by induction one sees that

$$\partial^{(\overline{k})} I_0 f = I_0 f^{(\overline{k})} + \sum_{l=0}^{\overline{k}_1 - 1} p(D')^{\overline{k}_1 - 1 - l} \partial^{(l, \overline{k}')} f\,, \tag{6.59}$$

where $f^{(\overline{k})} = p(D')^{\overline{k}_1} \partial^{(0,\overline{k}')} f$. Note that for $\overline{\beta} \in \{\eta + \theta, \gamma + \theta - \varepsilon\}$ there is a $\beta \in ([\eta, \gamma] \backslash A_{\mathbb{N}^d}) \cap (0, \gamma]$ such that $\overline{\beta} \leqslant \beta + \theta - \varepsilon$, more precisely there are two distinct cases we consider

1. There is a $\beta \in \{\eta + \varepsilon, \gamma\} \cap (0, \gamma]$ such that $\overline{\beta} = \beta + \theta - \varepsilon$,
2. $\eta < 0$ and $\overline{\beta} = \eta + \theta$, in which case we choose $\beta = \varepsilon$.

For the contribution of the second term in (6.59) we only have to consider case 1, indeed: If $\overline{\beta} = \eta + \theta$ with $\eta < 0$ we must have by definition of $\overline{k}$ that $\overline{k}_1 = 0$ and the second term in (6.59) equals 0. In case 1 we bound the latter, using $f \in \mathcal{C}^{\beta}_{\mathfrak{s}}(\Omega_t^T)$, in $\mathcal{C}^{\beta - |\overline{k}|_{\mathfrak{s}} + \theta}_{\mathfrak{s}}(\Omega_t^T) \subseteq \mathcal{C}^{\overline{\beta} - |\overline{k}|_{\mathfrak{s}}}_{\mathfrak{s}}(\Omega_t^T)$ by $t^{\frac{\eta - \beta}{\theta}} = t^{\frac{\eta + \theta - (\overline{\beta} + \varepsilon)}{\theta}} = t^{\frac{\eta + \theta - \varepsilon - \overline{\beta}}{\theta}}$.

It thus remains to consider the contribution of the first term $I_0 f^{(\overline{k})}$ in (6.59) to (6.58). We split

$$(I_0 f^{(\overline{k})})_y - (I_0 f^{(\overline{k})})_x = (I_0 f^{(\overline{k})})_{(y_1, y')} - (I_0 f^{(\overline{k})})_{(y_1, x')} + (I_0 f^{(\overline{k})})_{(y_1, x')} - (I_0 f^{(\overline{k})})_{(x_1, x')} \tag{6.60}$$

The first term can be estimated with $\beta \geqslant \overline{\beta} - \theta + \varepsilon$ as above and Lemma 6.3.2 by (writing $P_s := e^{sp(D')}$)

$$\begin{aligned} \int_0^{y_1} \mathrm{d}s\, &|(P_{y_1 - s} f^{(\overline{k})}(s))(y') - (P_{y_1 - s} f^{(\overline{k})}(s))(x')| \\ &\lesssim \int_0^{y_1} \mathrm{d}s\, (y_1 - s)^{\frac{(\beta - |\overline{k}|_{\mathfrak{s}}) - (\overline{\beta} - |\overline{k}|_{\mathfrak{s}})}{\theta}} s^{\frac{\eta - \beta}{\theta}} \|y - x\|_{\mathfrak{s}}^{\overline{\beta} - |\overline{k}|_{\mathfrak{s}}} \\ &\lesssim y_1^{\frac{\eta + \theta - \overline{\beta}}{\theta}} \|y - x\|_{\mathfrak{s}}^{\overline{\beta} - |\overline{k}|_{\mathfrak{s}}} \leqslant t^{\frac{\eta + \theta - \overline{\beta}}{\theta}} \|y - x\|_{\mathfrak{s}}^{\overline{\beta} - |\overline{k}|_{\mathfrak{s}}} \\ &\leqslant t^{\frac{\eta + \theta - \varepsilon - \overline{\beta}}{\theta}} \|y - x\|_{\mathfrak{s}}^{\overline{\beta} - |\overline{k}|_{\mathfrak{s}}}\,. \end{aligned}$$

For the second term in (6.60) we assume without loss of generality that $x_1 \leqslant y_1$ and split

$$\int_0^{x_1} \mathrm{d}s\, ((P_{y_1 - s} f^{((\overline{k}))}(s))(x') - (P_{x_1 - s} f^{(\overline{k})}(s))(x')) + \int_{x_1}^{y_1} \mathrm{d}s\, (P_{y_1 - s} f^{(\overline{k})}(s))(x') \tag{6.61}$$

The first term can be bounded precisely as above so that we only have to deal with the second term. Consider case 1 first. By definition of $\overline{k}$ we have for $\overline{\beta}-\theta \in \{\eta, \gamma-\varepsilon\}$ that $\overline{\beta} - \theta - |k|_{\mathfrak{s}} < 0$ and by our assumption on ε we still have for $\beta = \overline{\beta} - \theta + \varepsilon \in \{\eta + \varepsilon, \gamma\}$ that $\beta - |\overline{k}|_{\mathfrak{s}} < 0$. Consequently we can bound the second term in (6.61) via (6.53) in the proof of Lemma 6.3.2

$$\begin{aligned}
&\int_{x_1}^{y_1} \mathrm{d}s\, (y_1 - s)^{\frac{\beta-|k|_{\mathfrak{s}}}{\theta}} s^{\frac{\eta-\beta}{\theta}} = \int_{x_1}^{y_1} \mathrm{d}s\, (y_1 - s)^{\frac{\overline{\beta}-|k|_{\mathfrak{s}}-\theta+\varepsilon}{\theta}} s^{\frac{\eta-\beta}{\theta}} \\
&\leqslant (y_1 - x_1)^{\frac{\overline{\beta}-|k|_{\mathfrak{s}}}{\theta}} \int_{x_1}^{y_1} \mathrm{d}s\, (y_1 - s)^{\frac{-\theta+\varepsilon}{\theta}} s^{\frac{\eta-\beta}{\theta}} \\
&\leqslant (y_1 - x_1)^{\frac{\overline{\beta}-|k|_{\mathfrak{s}}}{\theta}} \int_{0}^{y_1} \mathrm{d}s\, (y_1 - s)^{\frac{-\theta+\varepsilon}{\theta}} s^{\frac{\eta-\beta}{\theta}} \lesssim (y_1 - x_1)^{\frac{\overline{\beta}-|k|_{\mathfrak{s}}}{\theta}} t^{\frac{\eta+\theta-\overline{\beta}}{\theta}} \\
&\leqslant \|y - x\|_{\mathfrak{s}}^{\overline{\beta}-|k|_{\mathfrak{s}}} t^{\frac{\eta+\theta-\varepsilon-\overline{\beta}}{\theta}} .
\end{aligned}$$

Considering case 2, note that by definition of $\overline{k}$ we must have $|\overline{k}| \in [0, \theta)$. If $\varepsilon - |\overline{k}|_{\mathfrak{s}} < 0$ we use once more (6.53) from the proof of Lemma 6.3.2 to estimate the second term in (6.61) by

$$\begin{aligned}
&\int_{x_1}^{y_1} \mathrm{d}s\, (y_1 - s)^{\frac{\varepsilon-|\overline{k}|_{\mathfrak{s}}}{\theta}} s^{\frac{\eta-\varepsilon}{\theta}} = \int_{x_1}^{y_1} \mathrm{d}s\, (y_1 - s)^{\frac{\varepsilon-(\eta+\theta)+\overline{\beta}-|k|_{\mathfrak{s}}}{\theta}} s^{\frac{\eta-\varepsilon}{\theta}} \\
&\leqslant (y_1 - x_1)^{\frac{\overline{\beta}-|\overline{k}|_{\mathfrak{s}}}{\theta}} \int_{x_1}^{y_1} \mathrm{d}s\, (y_1 - s)^{\frac{\varepsilon-(\eta+\theta)}{\theta}} s^{\frac{\eta-\varepsilon}{\theta}} \\
&\leqslant \|y - x\|^{\overline{\beta}-|\overline{k}|_{\mathfrak{s}}} \int_{0}^{y_1} \mathrm{d}s\, (y_1 - s)^{\frac{\varepsilon-(\eta+\theta)}{\theta}} s^{\frac{\eta-\varepsilon}{\theta}} \lesssim \|y - x\|^{\overline{\beta}-|\overline{k}|_{\mathfrak{s}}} \leqslant t^{-\frac{\varepsilon}{\theta}} \|y - x\|^{\overline{\beta}-|\overline{k}|_{\mathfrak{s}}} .
\end{aligned}$$

If $\varepsilon - |\overline{k}|_{\mathfrak{s}} > 0$ we use $\mathcal{C}_{\mathfrak{s}}^{\varepsilon-|\overline{k}|_{\mathfrak{s}}}(\Omega_t^T) \subseteq C_b(\Omega_t^T)$ to bound the second term in (6.61) by

$$\begin{aligned}
\int_{x_1}^{y_1} \mathrm{d}s\, s^{\frac{\eta-\varepsilon}{\theta}} \lesssim y_1^{\frac{\theta+\eta-\varepsilon}{\theta}} - x_1^{\frac{\theta+\eta-\varepsilon}{\theta}} &\leqslant t^{-\frac{\varepsilon}{\theta}} (y_1^{\frac{\theta+\eta}{\theta}} - x_1^{\frac{\theta+\eta}{\theta}}) \leqslant t^{-\frac{\varepsilon}{\theta}} (y_1 - x_1)^{\frac{\theta+\eta}{\theta}} \\
&\leqslant t^{-\frac{\varepsilon}{\theta}} \|y_1 - x_1\|_{\mathfrak{s}}^{\frac{\theta+\eta}{\theta}} ,
\end{aligned}$$

where we used that for $0 < b < a$, $\nu \in (0, 1)$ one has $a^\nu \leqslant (a - b)^\nu + b^\nu$ and thus $a^\nu - b^\nu \leqslant (a - b)^\nu$. □

Lemma 6.3.4. *Given a regularity structure $\mathscr{T} = (A, \mathcal{T}, G)$ satisfying Assumption 2.3.12 together with a model (Π, Γ) and a sector $\mathcal{V} \subseteq \mathcal{T}$ define for $\gamma \in \mathbb{R}$, $\alpha \in A$ with $\alpha < \gamma$ and $F \in \mathcal{D}^\gamma(\mathbb{R}^d; \mathcal{V} \backslash \overline{\mathcal{T}})$*

$$P^{(\nu,\tilde{\nu})}(F, \Gamma^\alpha)_x = \sum_{j>0} \iint \mathrm{d}u \mathrm{d}v\, \partial^\nu \Psi^{<j-1}_{x-u} \partial^{\tilde{\nu}} \Psi^j_{x-v}\, \Gamma_{vu} F_u \tag{6.62}$$

where $x \in \mathbb{R}^d$ and $\nu, \tilde{\nu} \in \mathbb{N}^d$, $\nu \neq 0$. It then holds

$$\|P^{(\nu,\tilde{\nu})}(F, \Gamma^\alpha)\|_{\mathcal{C}_\mathfrak{s}^{\gamma-\alpha-|\nu+\tilde{\nu}|_\mathfrak{s}}(\mathbb{R}^d;\mathbb{R})} \lesssim (1 + \|\Gamma\|_\gamma)\, \|F\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\overline{\mathcal{T}})} \,. \tag{6.63}$$

Given a second model $(\hat{\Pi}, \hat{\Gamma})$ and some $\hat{F} \in \mathcal{D}^\gamma(\mathbb{R}^d; \mathcal{V}\backslash\overline{\mathcal{T}}, \hat{\Gamma})$ we further have

$$\|P^{(\nu,\tilde{\nu})}(F, \Gamma^\alpha) - P^{(\nu,\tilde{\nu})}(\hat{F}, \hat{\Gamma}^\alpha)\|_{\mathcal{C}_\mathfrak{s}^{\gamma-\alpha-|\nu+\tilde{\nu}|_\mathfrak{s}}(\mathbb{R}^d;\mathbb{R})} \lesssim \\ (1 + \|\Gamma\|_\gamma)\|F; \hat{F}\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\overline{\mathcal{T}},\Gamma,\hat{\Gamma})} + \|\hat{F}\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\overline{\mathcal{T}},\hat{\Gamma})}\|\Gamma - \hat{\Gamma}\|_\gamma \,. \tag{6.64}$$

Remark 6.3.5. *If $|\nu + \tilde{\nu}|_\mathfrak{s} < \gamma - \alpha$ definition (6.62) should be read in distributional sense.*

Proof. We only show (6.63), estimate (6.64) follows by essentially the same arguments. Since the terms of (6.63) are spectrally supported in a rectangular annulus of size $2^{j\mathfrak{s}}\mathcal{A}$, it is sufficient with Lemma 2.1.19 to bound each term. Using that $\int \partial^{\tilde{\nu}} \Psi^{<j-1}_{x-u} = 0$ due to $\tilde{\nu} \neq 0$ and that polynomials in v vanish when integrated against Ψ^j_{x-v} we can reshape

$$\begin{aligned}
&\iint \mathrm{d}u\mathrm{d}v\, \partial^\nu \Psi^{<j-1}_{x-u} \partial^{\tilde{\nu}} \Psi^j_{x-v}\, \Gamma^\alpha_{vu} F_u = \iint \mathrm{d}u\mathrm{d}v\, \partial^\nu \Psi^{<j-1}_{x-u} \partial^{\tilde{\nu}} \Psi^j_{x-v}\, \Gamma^\alpha_{vu} (F_u - \Gamma_{ux} F_x) \\
&= \sum_{\alpha' \in A_{\mathcal{V}\backslash\overline{\mathcal{T}}}:\, \alpha' > \alpha} \iint \mathrm{d}u\mathrm{d}v\, \partial^\nu \Psi^{<j-1}_{x-u} \partial^{\tilde{\nu}} \Psi^j_{x-v}\, \Gamma^{\alpha'}_{vu} (F_u - \Gamma^{\alpha'}_{ux} F_x) \,,
\end{aligned}$$

which can be bounded via Lemma 2.1.14 by $2^{-j(\gamma-\alpha-|\nu+\tilde{\nu}|_\mathfrak{s})}(1 + \|\Gamma\|_\gamma)\, \|F\|_{\mathcal{D}^\gamma(\mathbb{R}^d;\mathcal{V}\backslash\overline{\mathcal{T}})}$, from which the claim follows. □

Lemma 6.3.6. *Given a regularity structure $\mathscr{T} = (A, \mathcal{T}, G)$ together with a model (Π, Γ) and a function-like sector $\mathcal{V} \subseteq \mathcal{T}$, we have for $\tau \in \mathcal{V}$ that $\Pi_x \tau$ for $x \in \mathbb{R}^d$ can be identified with a continuous function, namely*

$$\Pi_x \tau(y) = \Gamma^{\mathbf{1}}_{yx} \tau$$

Proof. We proceed similar as for the uniqueness part in [Hai14, Theorem 3.10]. Take a mollifier sequence ρ^ε, i.e. choose $\rho \in C^\infty_c(\mathbb{R}^d, \mathbb{R})$ with $\int \mathrm{d}x\, \rho(x) = 1$ and set $\rho^\varepsilon(x) = \varepsilon^{-|\mathfrak{s}|}\rho(2^{-j\mathfrak{s}}x)$. One then sees for $z \in \mathbb{R}^d$

$$\begin{aligned}
\lim_{\varepsilon \to 0} (\Pi_x \tau - \Gamma^{\mathbf{1}}_{\cdot x})(\rho^\varepsilon(\cdot - z)) &= \Gamma^{\mathbf{1}}_{zx}\tau + \lim_{\varepsilon \to 0} \sum_{\alpha \in A_\mathcal{V},\, \alpha > 0} (\Pi_z \Gamma^\alpha_{zx} \tau - \Gamma^{\mathbf{1}}_{\cdot x})(\rho^\varepsilon(\cdot - z)) \\
&= \Gamma^{\mathbf{1}}_{zx}\tau - \Gamma^{\mathbf{1}}_{zx}\tau = 0 \,,
\end{aligned}$$

Taking then a general $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and we have $(\Pi_x \tau - \Gamma^{\mathbf{1}}_{\cdot x})(\varphi) = \lim_{\varepsilon \to 0}(\Pi_x \tau - \Gamma^{\mathbf{1}}_{\cdot x})(\rho^\varepsilon * \varphi) = \int \mathrm{d}z\, \varphi(z) \lim_{\varepsilon \to 0}(\Pi_x \tau - \Gamma^{\mathbf{1}}_{\cdot x})(\rho^\varepsilon(\cdot - z)) = 0$, which proves the claim. □

Lemma 6.3.7. *Let $\mathscr{T} = (A, \mathcal{T}, G)$ be a regularity structure together with a model (Π, Γ) and satisfying Assumption 2.3.12. Let $\eta, \gamma \in (0,\infty)\backslash A_{\mathbb{N}^d}$ with $\eta \leqslant \gamma$, and let $\mathcal{V} \subseteq \mathcal{T}$ be a sector such that $\mathcal{V}\backslash\overline{\mathcal{T}}$ has regularity $\alpha_0 > 0$. Let further $F \in \mathcal{D}^{[\eta,\gamma]}(\Omega^T; \mathcal{V}\backslash\overline{\mathcal{T}})$.*

We have for any $\kappa > 0$ and $\beta \in [\eta,\gamma]\backslash A_{\mathbb{N}^d}$ with $\beta - \big(\alpha_0 \wedge (\eta-\kappa)\big) < \theta$ that $P(\overline{F^{<\beta}}, \Gamma^{\mathbf{1}})|_{\{x_1=0\}} \in \mathcal{C}^{(\eta-\kappa)\wedge\alpha_0}(\mathbb{R}^{d-1})$ (with isotropic scaling), where $\overline{F^{<\beta}}$ denotes the poor man's extension from Subsection 5.3.2. Moreover, for $p \in S^\theta(\mathbb{R}^{d-1})$ and $t \in (0,T)$ it holds

$$\left\|(s,x') \mapsto e^{-sp(D')}\left(P(\overline{F^{<\beta}}, \Gamma^{\mathbf{1}})|_{\{x_1=0\}}\right)(x')\right\|_{\mathcal{C}^\beta_{\mathfrak{s}}(\Omega^T_t)} \lesssim t^{\frac{(\eta-\kappa)\wedge\alpha_0-\beta}{\theta}} \|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\overline{\mathcal{T}})}\,. \tag{6.65}$$

Given a second model $(\hat{\Pi}, \hat{\Gamma})$ and $\hat{F} \in \mathcal{D}^{[\eta,\gamma]}(\Omega^T; \mathcal{V}\backslash\overline{\mathcal{T}}, \hat{\Gamma})$ one further has

$$\begin{aligned}&\left\|(s,x') \mapsto e^{-sp(D')}\left(P(\overline{F^{<\beta}}, \Gamma^{\mathbf{1}})|_{\{x_1=0\}} - P(\overline{\hat{F}^{<\beta}}, \hat{\Gamma}^{\mathbf{1}})|_{\{x_1=0\}}\right)(x')\right\|_{\mathcal{C}^\beta_{\mathfrak{s}}(\Omega^T_t)} \lesssim t^{\frac{(\eta-\kappa)\wedge\alpha_0-\beta}{\theta}} \\ &\times \left(\|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\overline{\mathcal{T}},\Gamma)} \|\Gamma-\hat{\Gamma}\|_\gamma + \|\hat{\Gamma}\|_\gamma \|F;\hat{F}\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\overline{\mathcal{T}},\Gamma,\hat{\Gamma})}\right).\end{aligned} \tag{6.66}$$

Proof. We consider the estimate (6.65), (6.66) follows essentially by the same arguments. We can assume without loss of generality that $\|F\|_{\mathcal{D}^{[\eta,\gamma]}(\Omega^T;\mathcal{V}\backslash\overline{\mathcal{T}})} \leqslant 1$ and that $\kappa > 0$ is so small that $\eta - \kappa > 0$. Let us start with the following remark: Suppose $\psi \in \mathcal{S}(\mathbb{R}^d)$ is such that $\mathcal{F}_{\mathbb{R}^d}\psi \subseteq \{\xi \in \mathbb{R}^d \,|\, \|\xi\|_\infty \leqslant C\}$ for some $C > 0$, we then still have for any $x_1 \in \mathbb{R}$ $\operatorname{supp}\mathcal{F}_{\mathbb{R}^{d-1}}(\psi(x_1,\cdot)) \subseteq \{\xi' \in \mathbb{R}^{d-1} \,|\, \|\xi'\|_\infty \leqslant C\}$, as can be seen by writing $\mathcal{F}_{\mathbb{R}^{d-1}}(\psi(x_1,\cdot))(\xi') = \int_{\mathbb{R}} \mathrm{d}\xi_1\, e^{2\pi\imath x_1\xi_1}\, \mathcal{F}_{\mathbb{R}^d}\psi(\xi_1,\xi')$.

Consequently, we have for any $u, v \in \mathbb{R}^d$ that $x' \mapsto \Psi^{<j-1}_{(0,x')-u}\Psi^j_{(0,x')-v}$ is spectrally supported in a box of size $2^j\mathcal{B} \subseteq \mathbb{R}^{d-1}$ and so are the terms of

$$\begin{aligned}P(\overline{F^{<\beta}}, \Gamma^{\mathbf{1}})_{(0,x')} &= \sum_{j>0}\iint \mathrm{d}u\mathrm{d}v\, \Psi^{<j-1}_{(0,x')-u}\Psi^j_{(0,x')-v}\Gamma^{\mathbf{1}}_{v\overline{u}}F^{<\beta}_{\overline{u}} \\ &= \sum_{A_{\mathcal{V}\backslash\overline{\mathcal{T}}}:\,\alpha<\beta}\Big[\sum_{j>0}\iint \mathrm{d}u\mathrm{d}v\, \Psi^{<j-1}_{(0,x')-u}\Psi^j_{(0,x')-v}\mathbf{1}_{|u_1|<T/2}\Gamma^{\mathbf{1}}_{v\overline{u}}F^{\alpha}_{\overline{u}} \\ &+ \iint \mathrm{d}u\mathrm{d}v\, \Psi^{<j-1}_{(0,x')-u}\Psi^j_{(0,x')-v}\mathbf{1}_{|u_1|\geqslant T/2}\Gamma^{\mathbf{1}}_{v\overline{u}}F^{\alpha}_{\overline{u}}\Big] \\ &=: \sum_{\alpha\in A_{\mathcal{V}\backslash\overline{\mathcal{T}}}:\,\alpha<\beta}\sum_{j>0}[f^\alpha_{1,j}(x') + f^\alpha_{2,j}(x')] =: \sum_{\alpha\in A_{\mathcal{V}\backslash\overline{\mathcal{T}}}:\,\alpha<\beta}[f^\alpha_1(x') + f^\alpha_2(x')]\,,\end{aligned}$$

where we used the fact that polynomial entries of F vanish (Lemma 2.1.14) and where $\|f^\alpha_1\|_{\mathcal{C}^{(\eta-\kappa)\wedge\alpha}(\mathbb{R}^{d-1})} \lesssim 1$, $\|f^\alpha_2\|_{\mathcal{C}^\alpha(\mathbb{R}^{d-1})} \lesssim T^{\frac{(\eta-\kappa-\alpha)\wedge 0}{\theta}}$, which follows from Lemma

2.1.19 and the following bounds

$$\|f_{1,j}^{\alpha}(x')\|_{\mathcal{T}_\alpha} \lesssim \iint \mathrm{d}u\mathrm{d}v\, |\Psi^{<j-1}_{(0,x')-u}\Psi^{j}_{(0,x')-v}|\, \|\overline{u}-v\|_{\mathfrak{s}}^{\alpha}\, |u_1|^{\frac{(\eta-\alpha-\kappa)\wedge 0}{\theta}} \lesssim 2^{-j(\alpha+(\alpha-\eta-\kappa)\wedge 0)}\,,$$

$$\|f_{2,j}^{\alpha}(x')\|_{\mathcal{T}_\alpha} \lesssim \iint \mathrm{d}u\mathrm{d}v\, |\Psi^{<j-1}_{(0,x')-u}\Psi^{j}_{(0,x')-v}|\, \|\overline{u}-v\|^{\alpha}\, T^{\frac{(\alpha-\eta-\kappa)\wedge 0}{\theta}} \lesssim T^{\frac{(\alpha-\eta-\kappa)\wedge 0}{\theta}}\, 2^{-j\alpha}\,.$$

In the first step we used Lemma 5.3.5 and dropped the indicator function thereafter. For the second step we used the inequality $\|\overline{u}-v\|_{\mathfrak{s}}^{\alpha} \lesssim \|\overline{u}-(0,x')\|_{\mathfrak{s}}^{\alpha}+\|v-(0,x')\|_{\mathfrak{s}}^{\alpha} \leqslant \|u-(0,x')\|_{\mathfrak{s}}^{\alpha}+\|v-(0,x')\|_{\mathfrak{s}}^{\alpha}($ due to $|\overline{u}_1| \leqslant |u_1|)$ and Lemma 2.1.14.

We see therefore that indeed $P(\overline{F^{<\beta}},\Gamma^{\mathbf{1}}) \in \mathcal{C}^{(\eta-\kappa)\wedge\alpha_0}(\mathbb{R}^{d-1})$ and by the regularizing property of the semigroup $e^{-sp(D')}$ from Lemma 6.3.2 we obtain

$$\begin{aligned}\|(s,x')\mapsto e^{-sp(D')}P(\overline{F^{<\beta}},\Gamma^{\mathbf{1}})(x')\|_{\mathcal{C}_{\mathfrak{s}}^{\beta}(\Omega_t^T)} &\lesssim \sum_{\alpha\in A_{\mathcal{V}\setminus\overline{\mathcal{T}}}\colon \alpha<\beta} \left(t^{\frac{(\eta-\kappa)\wedge\alpha-\beta}{\theta}} + T^{\frac{(\eta-\kappa-\alpha)\wedge 0}{\theta}}\, t^{\frac{\alpha-\beta}{\theta}}\right)\\ &\leqslant 2 \sum_{\alpha\in A_{\mathcal{V}\setminus\overline{\mathcal{T}}}\colon \alpha<\beta} t^{\frac{(\eta-\kappa)\wedge\alpha-\beta}{\theta}}\,,\end{aligned}$$

where we used $(\eta-\kappa-\alpha)\wedge 0 = (\eta-\kappa)\wedge\alpha-\alpha$ and $T \geqslant t$ in the last step. Since the terms on the right hand side can be bounded up to a constant by $t^{\frac{(\eta-\kappa)\wedge\alpha_0-\beta}{\theta}}$, we have shown (6.65). □

Chapter 7

Applying regularity structures to option pricing

A rough volatility model is a generalization of the well-known Black-Scholes model [Bla76] which describes the evolution of an asset $(S_t)_{t\in[0,T]}$ according to

$$\mathrm{d}S_t = S_t \cdot \sigma\,\mathrm{d}B_t\,, \tag{7.1}$$

where B_t is a Brownian motion and the parameter $\sigma > 0$ is known as the *volatility*. The price for a call option on (7.1) with strike price K and expiry date $T > 0$ should be chosen as

$$C_{\text{B.S.}}(S_0,\, K,\, \sigma^2 T) := \mathbb{E}[(S_T - K)_+] = \mathbb{E}\left[\left(S_0\,\exp(\sigma B_T - \sigma^2 T/2) - K\right)_+\right]\,, \tag{7.2}$$

where we used $\sigma B_T \sim \mathcal{N}(0, \sigma^2 T)$ to describe the right hand side really as a function of only three deterministic variables. It seems natural to generalize (7.1) to a system where σ is allowed to be a stochastic process $(\sigma_t)_{t\in[0,T]}$. An idea that goes back to Heston [Hes93] is to split $B = \rho W + \sqrt{1-\rho^2}\,\overline{W}$ into two independent Wiener processes W, $\overline{W}$ with some correlation parameter $\rho \in [0,1]$ and to consider σ driven by W. Gatheral, Jaisson and Rosenbaum [GJR17] observed that σ behaves, at least for time scales which are not too big, like the (stochastic) exponential of a *fractional Brownian motion* with a Hurst index around $H \sim 0.1$. Recalling that Brownian motion has Hurst index $H = 1/2$, we see that volatiliy is quite "rough". To take this fact into account we here consider the model proposed in [BFG16] (the *rough Bergomi model*), which assumes that $(\sigma_t)_{t\in[0,T]}$ is of the form

$$\sigma_t = f(\hat{W}_t, t) \tag{7.3}$$

where f in [BFG16] is actually chosen to be a (stochastic) exponential in the process $\hat{W}_t$, which is a Riemann-Liouville fractional Brownian motion

$$\hat{W}_t = \int_0^t K(t-r)\,\mathrm{d}W_r$$

with the *Volterra* kernel $K(r) = \sqrt{2H}\mathbf{1}_{r>0}\, r^{H-1/2}$ and W a Brownian motion as in the Heston model. Let us summarize the total model as proposed in [BFG16]:

$$\begin{aligned} \mathrm{d}S_t &= S_t \cdot f(\hat{W}_t, t)\, \mathrm{d}(\rho W + \sqrt{1-\rho^2}\, \overline{W})_t \,, \\ \hat{W}_t &= \int_0^t K(t-r)\, \mathrm{d}W_r \,. \end{aligned}$$

By conditioning first on W (a trick that goes back to [RT97]) one can derive a rather explicit formula for the corresponding option price, namely

$$\mathbb{E}[(S_T - K)_+] = \mathbb{E}\left[\mathbb{E}[(S_T-K)_+ | (W_t)_{t\in[0,T]}]\right]$$

$$= \mathbb{E}\Big[\mathbb{E}\Big[(S_0 e^{\rho \int_0^T f(\hat{W}_t,t)\mathrm{d}W_t - \frac{\rho^2}{2}\int_0^T f(\hat{W}_t,t)^2 \mathrm{d}t} \tag{7.4}$$

$$\times\, e^{\sqrt{1-\rho^2}\int_0^T f(\hat{W}_t,t)\,\mathrm{d}\overline{W}_t - \frac{1-\rho^2}{2}\int_0^T f(\hat{W}_t,t)^2\,\mathrm{d}t} - K)_+ \Big| (W_t)_{t\in[0,T]}\Big]\Big]$$

$$= \mathbb{E}\Big[C_{\mathrm{B.S.}}\Big(S_0 \exp(\rho \int_0^T f(\hat{W}_t,t)\,\mathrm{d}W_t - \frac{\rho^2}{2}\int_0^T f(\hat{W}_t,t)^2 \mathrm{d}t\,, \tag{7.5}$$

$$K\,,\ \frac{1-\rho^2}{2}\int_0^T f(\hat{W}_t,t)^2\,\mathrm{d}t\Big)\Big] \tag{7.6}$$

where we used in the last step that, conditioned on $(W_t)_{t\in[0,T]}$, one has

$$\sqrt{1-\rho^2}\int_0^T f(\hat{W}_t,t)\,\mathrm{d}\overline{W}_t \overset{\mathrm{d}}{\sim} \mathcal{N}\Big(0, (1-\rho^2)\int_0^T f(\hat{W}_t,t)^2\mathrm{d}t\Big)$$

and where $C_{\mathrm{B.S.}}$ is defined as in (7.2). As explained in the introduction of this thesis the critical term in (7.6) is the Itô integral

$$\int_0^T f(\hat{W}_t,t)\,\mathrm{d}W_t \tag{7.7}$$

since for a given approximation $(W_t^\varepsilon)_{t\in[0,T]}$ of W and $\hat{W}_t^\varepsilon := \int_0^T K(t-r)\,\mathrm{d}W^\varepsilon(t)$ one expects via the Wong-Zakai [WZ65] convergence result

$$\int_0^T f(\hat{W}_t^\varepsilon,t)\,\mathrm{d}W_t^\varepsilon \to \int_0^T f(\hat{W}_t,t)\,\mathrm{d}W_t + c(T) \tag{7.8}$$

where $c(T)$ is some Itô-Stratonovich correction. For $H < 1/2$ one sees by scaling that for non-trivial f the correction $c(T)$ is non-existent, so that it is not a priori clear how one could approximate (7.8) and thus (7.6). We here develop an approximation theory for integrals of the type (7.7) by using the theory of regularity structures, which we recalled in Chapter 2.

The content of this chapter is essentially taken from [BFG$^+$17] and we refer to this article for a more thorough discussion of the application to option pricing and for numerical results.

The key idea in this chapter is to replace (7.8) by a rigoruous convergence result, namely

$$\int_0^T f(\hat{W}_t^\varepsilon, t)\,\mathrm{d}W_t^\varepsilon - \int_0^T \mathscr{C}^\varepsilon(t)\partial_1 f(\hat{W}_t^\varepsilon, t)\,\mathrm{d}t \to \int_0^T f(\hat{W}_t, t)\,\mathrm{d}W_t\,, \tag{7.9}$$

where $\mathscr{C}^\varepsilon(t)$ denotes some function which diverges for $\varepsilon \to 0$. The extra term

$$-\int_0^T \mathscr{C}^\varepsilon(t)\partial_1 f(\hat{W}_t^\varepsilon, t)\,\mathrm{d}t$$

can thus be seen as some cancellation that erases the diverging Itô-Stratonovich correction from (7.8). Our main result is stated in Theorem 7.2.9. In Section 7.1 we develop the regularity structure on which we describe integrals of the type (7.7) and their approximations via different models. In Section 7.2 we use the Reconstruction Theorem 2.3.19 to derive (7.9) (together with a rate). As the theory is completely one-dimensional there is no need of a scaling vector $\mathfrak{s}$ as introduced in Chapter 2 or, in other words, we simply take $\mathfrak{s} = 1$.

7.1 Regularity structure and models

Definition of the considered regularity structure

We first build a regularity structure which allows us to describe products between the derivative of a Brownian motion W and powers of a fractional Brownian motion $\hat{W}$, constructed from W. Fix a parameter $H \in (0, 1/2]$, which we will identify below with the Hurst index of $\hat{W}$, an arbitrary $\kappa \in (0, H)$ and an integer

$$M \geqslant \max\{m \in \mathbb{N} \,|\, m \cdot (H - \kappa) - 1/2 - \kappa \leqslant 0\}$$

so that

$$(M + 1)(H - \kappa) - 1/2 - \kappa > 0\,. \tag{7.10}$$

We introduce now a regularity structure geneterated by a set of (abstract) symbols

$$S = S^{(M)} = \left\{\Xi, \Xi\mathcal{I}(\Xi), \dots, \Xi\mathcal{I}(\Xi)^M, \mathbf{1}, \mathcal{I}(\Xi), \dots, \mathcal{I}(\Xi)^M\right\}\,, \tag{7.11}$$

so that

$$\mathcal{T} = \mathcal{T}^{(M)} = \mathrm{span}\,\{S\} = \bigoplus_{\tau \in S} \mathbb{R}\tau\,.$$

The interpretation for the symbols in S is as follows: Ξ should be understood as an abstract representation of the white noise ξ belonging to the Brownian motion W, i.e. $\xi = \dot{W}$ where the derivative is taken in the distributional sense. We will extend W (and thus ξ) by 0 on the negative axis. The symbol $\mathcal{I}(\dots)$ has in this chapter the intuitive meaning "integration against the Volterra kernel", so that $\mathcal{I}(\Xi)$ represents the integration of white noise against the Volterra kernel

$$\sqrt{2H}\int_0^t |t-r|^{H-1/2}\mathrm{d}W_r\,,$$

which is nothing but the fractional Brownian motion $\hat{W}_t$. Symbols like $\Xi\mathcal{I}(\Xi)^m = \Xi\cdot\mathcal{I}(\Xi)\cdot\ldots\cdot\mathcal{I}(\Xi)$ or $\mathcal{I}(\Xi)^m = \mathcal{I}(\Xi)\cdot\ldots\cdot\mathcal{I}(\Xi)$ should be read as products between the objects above. These interpretations of the symbols generating $\mathcal{T}$ will be made rigorous by the model (Π,Γ) in the next subsection. Every symbol in $\tau\in S$ is assigned a homogeneity $|\tau|$, which we define by

$$\begin{aligned}|\Xi\mathcal{I}(\Xi)^m| &= -1/2-\kappa+m(H-\kappa),\ m\geqslant 0\\ |\mathcal{I}(\Xi)^m| &= m(H-\kappa),\ m>0\\ |\mathbf{1}| &= 0\,,\end{aligned}$$

We collect the homogeneities of elements of S in the set $A := \{|\tau|\,|\,\tau\in S\}$, whose minimum is $|\Xi| = -1/2-\kappa$. Note that the homogeneities are multiplicative in the sense that $|\tau\cdot\tau'| = |\tau|+|\tau'|$ for $\tau,\tau'\in S$.

At last, our regularity comes with a *structure group* G acting on the model space $\mathcal{T}$, which should satisfy $\Gamma\tau-\tau = \bigoplus_{\tau'\in S\colon|\tau'|<|\tau|}\mathbb{R}\tau'$ and $\Gamma\mathbf{1}=\mathbf{1}$ for $\tau\in S$ and $\Gamma\in G$. We will choose $G = \{\Gamma_h\,|\,h\in\mathbb{R}\}$ with the group law $\Gamma_h\Gamma_{h'} := \Gamma_{h+h'}$ for $h,\ h'\in\mathbb{R}$ (so that in fact $G\simeq(\mathbb{R},+)$). We define the action of G on $\mathcal{T}$ by

$$\Gamma_h\mathbf{1}=\mathbf{1},\ \Gamma_h\Xi=\Xi,\ \Gamma_h\mathcal{I}(\Xi)=\mathcal{I}(\Xi)+h\mathbf{1}\,.$$

and $\Gamma_h(\tau'\cdot\tau) = \Gamma_h\tau'\cdot\Gamma_h\tau$ for τ', $\tau\in S$ for which $\tau\cdot\tau'\in S$ is defined. The triple $\mathscr{T} := (A,\mathcal{T},G)$ is then a regularity structure as in Definition 2.3.2. We now equip $\mathscr{T}$ with a model as in Definition 2.3.9.

Models on $\mathscr{T}$

The limiting model $(\hat{\Pi},\hat{\Gamma})$

Throughout this chapter W is a Brownian motion on $\mathbb{R}_+$, which we extend to the negative axis by setting $W(x)=0$ for $x<0$. We will frequently use the notations

$$\int_0^t f(t)\mathrm{d}W_t,\ \int_0^t f(t)\diamond\mathrm{d}W_t \tag{7.12}$$

to denote the Itô integral and the Skorohod integral (which boils down to an Itô integral whenever the integrand is adapted).

Remark 7.1.1. *The same choice of the symbol "$\diamond$" in this thesis for Wick products and Skorohod integrals reflects a close relation between these objects. Indeed, when $g = \sum X_s \mathbf{1}_{[s,t]}$, with summation over a finite partition of $[0,T]$, and each X_s a (maybe non-adapted) random variable in a finite Wiener-Itô chaos, it follows from [Jan97, Thm 7.40] that $\int g \diamond \mathrm{d}W = \sum X_s \diamond (W_t - W_s)$. Passage to L^2-limits is then standard, so that the Skorohod integral is in a way the "integrated Wick product with $\frac{\mathrm{d}}{\mathrm{d}t}W$". We have already seen a variation of this fact in Section 3.4.*

From W we construct now the fractional Riemann-Liouville Brownian motion $\hat{W}$ with Hurst index $H \in (0, 1/2]$ as

$$\hat{W}(t) = (K * \dot{W})(t) = \sqrt{2H} \int_0^t |t-r|^{H-1/2} \, \mathrm{d}W_r \,,$$

where $K(t) = \sqrt{2H}\mathbf{1}_{t>0} \cdot t^{H-1/2}$ denotes the Volterra kernel. We also write $K(s,t) := K(t-s)$.

To give a meaning to the product terms $\Xi\mathcal{I}(\Xi)^k$ we follow the ideas from rough paths and define an "iterated integral" for $s, t \in \mathbb{R}, s \leqslant t$ as [1]

$$\mathbb{W}^m_{s,t} = \int_s^t (\hat{W}_r - \hat{W}_s)^m \, \mathrm{d}W_r \tag{7.13}$$

$\mathbb{W}^m(s,t)$ satisfies a modification of Chen's relation

Lemma 7.1.2. $\mathbb{W}^m$ *as defined in* (7.13) *satisfies*

$$\mathbb{W}^m_{s,t} = \mathbb{W}^m_{s,u} + \sum_{l=0}^m \binom{m}{l} (\hat{W}_u - \hat{W}_s)^l \mathbb{W}^{m-l}_{u,t} \tag{7.14}$$

for $s, u, t \in \mathbb{R}$, $s \leqslant u \leqslant t$.

Proof. Direct consequence of the binomial theorem. □

We extend the domain of $\mathbb{W}^m$ to all of $\mathbb{R}^2$ by imposing Chen's relation for all $s, u, t \in \mathbb{R}$, i.e. we set for $t, s \in \mathbb{R}$, $t \leqslant s$

$$\mathbb{W}^m_{s,t} = -\sum_{l=0}^m \binom{m}{l} (\hat{W}_t - \hat{W}_s)^l \mathbb{W}^{m-l}_{t,s} \tag{7.15}$$

[1] Some care is needed at this point: relation (7.13) only defines $\mathbb{W}^m_{s,t}$ up to some null set that might depend on s, t. Using the same arguments as for the brownian rough path [FH14] one sees however, via a Kolmogorov-like argument, that this null set can be chosen independent of s, t.

We are now in the position to define a model $(\hat\Pi, \hat\Gamma)$ in the sense of Definition 2.3.9 that gives a rigorous meaning to the interpretation we gave above for $\Xi, \mathcal{I}(\Xi), \Xi\mathcal{I}(\Xi), \ldots$. We define for $s,\, t \in \mathbb{R}$

$$
\begin{aligned}
&\hat\Pi_s \mathbf{1} = 1 && \hat\Gamma_{ts}\mathbf{1} = \mathbf{1}\\
&\hat\Pi_s \Xi = \tfrac{\mathrm{d}}{\mathrm{d}t} W = \dot W && \hat\Gamma_{ts}\Xi = \Xi\\
&\hat\Pi_s \mathcal{I}(\Xi)^m = \left(\hat W_\cdot - \hat W_s\right)^m && \hat\Gamma_{ts}\mathcal{I}(\Xi) = \mathcal{I}(\Xi) + (\hat W_t - \hat W_s)\mathbf{1}\\
&\hat\Pi_s \Xi\mathcal{I}(\Xi)^m = \tfrac{\mathrm{d}}{\mathrm{d}t}\mathbb{W}^m_{s,\cdot} && \hat\Gamma_{ts}\tau\tau' = \hat\Gamma_{ts}\tau \cdot \hat\Gamma_{ts}\tau'\,, \quad \text{for } \tau,\tau' \in S \text{ with } \tau\tau' \in S
\end{aligned}
$$

the derivative $\frac{\mathrm{d}}{\mathrm{d}t}$ is taken in the *distributional* sense. We extend both maps from S to $\mathcal{T}$ by imposing linearity. Note that $\hat\Gamma_{ts} = \Gamma_{\hat W_t - \hat W_s}$ by definition of our group $G = \{\Gamma_h \,|\, h \in \mathbb{R}\}$ above.

Lemma 7.1.3. *The pair* $(\hat\Pi, \hat\Gamma)$ *as defined above defines (a.s.) a model on* $(A, \mathcal{T}, G)$ *with local bounds.*

Remark 7.1.4. *The fact that* $(\hat\Pi, \hat\Gamma)$ *only has local bounds is in fact not a big issue. If one considers for example a Brownian motion* $W'_x := W_{x\wedge a}$ *that "freezes" for some large* $a > 0$ *we can replace* $(\hat\Pi, \hat\Gamma)$ *by a corresponding model* $(\hat\Pi', \hat\Gamma')$ *constructed from* W'*. The models* $(\hat\Pi, \hat\Gamma)$, $(\hat\Pi', \hat\Gamma')$ *are indistinguishable on* $(-\infty, a)$*, so that for example* $\hat\Pi_x(\varphi) = \hat\Pi'_x(\varphi)$ *for* $x < a$ *and* $\operatorname{supp}\varphi \subseteq (-\infty, a)$*. We will refer to this construction below, to allow for proofs with Fourier methods similar as in Chapter 5.*

Proof. The only symbols in S on which the relation $\hat\Pi_s\hat\Gamma_{st} = \hat\Pi_t$ from Definition 2.3.9 is not straightforward are those in the form $\Xi\mathcal{I}(\Xi)^m$, where the statement follows by Chen's relation. The bounds (2.38) and (2.39) follow for $\mathbf{1}$ trivially and for $\mathcal{I}(\Xi)^m$ by the $H-\kappa'$ Hölder regularity of $\hat W$ for $\kappa' \in (0, H)$. It is further straightforward to check the condition $\hat\Gamma_{st}\hat\Gamma_{tu} = \hat\Gamma_{su}$ from Definition 2.3.9 by using the rule $\hat\Gamma_{ts}\tau\tau' = \hat\Gamma_{ts}\tau \cdot \hat\Gamma_{ts}\tau'$ so that we are only left with the task to bound $\hat\Pi_s\Xi\mathcal{I}(\Xi)^m(\varphi^\lambda_{\cdot - s})$. Following along the lines of proof [FH14, Theorem 3.1] it follows $|\mathbb{W}^m_{s,t}| \leqslant K|s-t|^{mH+1/2-(m+1)\kappa}$ (where $K > 0$ denotes a random constant with $K \in \bigcap_{p<\infty} L^p$ that changes from line to line), so that

$$
\begin{aligned}
|\hat\Pi_s\mathcal{I}(\Xi)^m\Xi(\varphi^\lambda_{\cdot-s})| &= \left| \int \left(\varphi^\lambda\right)'(t-s)\mathbb{W}^m_{s,t}\mathrm{d}t \right| \\
&\leqslant K \int \varphi'(\lambda^{-1}(t-s))|s-t|^{mH+1/2-(m+1)\kappa}\frac{\mathrm{d}t}{\lambda^2}\\
&\leqslant K\lambda^{mH-1/2-(m+1)\kappa} = K\lambda^{|\mathcal{I}(\Xi)^m\Xi|}\,.
\end{aligned}
$$

□

As we will see below in Section 7.2 this model is the toolbox from which we can build pathwise Itô integrals of the type $\int_0^t f(r, \hat{W}_r)\,\mathrm{d}W_r$. For an approximation theory for such expressions we are in need of a comparable setup that describes approximations, which will be achieved by introducing a model $(\Pi^\varepsilon, \Gamma^\varepsilon)$.

The approximating model $(\Pi^\varepsilon, \Gamma^\varepsilon)$

The whole definition of the model (Π, Γ) is based on the object $\dot{W}$. It is therefore natural to build an approximating model by replacing $\dot{W}$ by some modification $\dot{W}^\varepsilon$ that converges (as a distribution) to $\dot{W}$ as $\varepsilon \to 0$.

The definition of $\dot{W}^\varepsilon$ will be based on a convolution with an object δ^ε which should be thought of as an approximation to the Dirac delta distribution. Our purpose to build δ^ε from wavelets, which can be as irregular as the Haar functions. We find it therefore convenient to allow δ^ε to take values in the Besov space $\mathcal{C}_1^\beta(\mathbb{R})$ for some $\beta > 1/2 + \kappa$, where $\mathcal{C}_1^\beta(\mathbb{R}) = \mathcal{B}_{1,\infty}^\beta(\mathbb{R}) = \mathcal{B}_{1,\infty}^\beta(\mathbb{R})$ is given as in Definition 2.1.16 with scaling now simply set to $\mathfrak{s} = 1$. This assumption covers Haar wavelets since we have $\mathbf{1}_{[0,1]} \in \mathcal{C}_1^1(\mathbb{R})$, which we prove in Section 7.4 below.

Definition 7.1.5. *In the following we call $\delta^\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ a measurable, bounded function with the following properties*

- $\delta^\varepsilon(x, y) = \delta^\varepsilon(y, x)$ *for all* $x, y \in \mathbb{R}$.
- *The map* $\mathbb{R} \ni x \mapsto \delta^\varepsilon(x, \cdot) \in \mathcal{C}_1^\beta(\mathbb{R})$ *is bounded and measurable for some* $\beta > -|\Xi| = 1/2 + \kappa$.
- *For* $y \in \mathbb{R}$ *one has* $\int_{\mathbb{R}} \delta^\varepsilon(x, y)\,\mathrm{d}x = 1$.
- $\sup_{\mathbb{R}^2} |\delta^\varepsilon| \lesssim \varepsilon^{-1}$.
- $\operatorname{supp} \delta^\varepsilon(x, \cdot) \subseteq B(x, c \cdot \varepsilon)$ *for any* $x \in \mathbb{R}$ *and some* $c > 0$ *(independent of* x*).*

Example 7.1.6. *There are two examples which are of particular interest for our purposes, for both we take* $\beta > 1/2 + \kappa$*:*

- *We say that* δ^ε *"comes from a mollifier", by which we mean that there is symmetric, compactly supported* $L^\infty(\mathbb{R}) \cap \mathcal{C}_1^\beta(\mathbb{R})$*-function* ρ*, which integrates to* 1 *such that*

$$\delta^\varepsilon(x, y) = \varepsilon^{-1} \cdot \rho(\varepsilon^{-1}(y - x))\,,$$

- *A further interesting example is the case where δ^ε "comes from a wavelet basis". Consider only $\varepsilon = 2^{-N}$ and choose compactly supported $L^\infty(\mathbb{R}) \cap \mathcal{C}_1^\beta(\mathbb{R})$-valued father wavelets $(\phi_{k,N})_{k\in\mathbb{Z}}$ (e.g. the Haar father wavelets $\phi_{k,N} = 2^{N/2} \cdot \mathbf{1}_{[k2^{-N},(k+1)2^{-N})}$) and set*

$$\delta^\varepsilon(x,y) = \sum_{k\in\mathbb{Z}} \phi_{k,N}(x)\phi_{k,N}(y)\,.$$

Note that (locally) $\dot{W}$ is contained in $\mathcal{C}^{|\Xi|}(\mathbb{R})$ (recall: $|\Xi| = -1/2-\kappa$). By Lemma 2.1.29 and 2.1.27 we have $\mathcal{C}^{|\Xi|}(\mathbb{R}) = \left(\mathcal{B}_{1,1}^{-|\Xi|}(\mathbb{R})\right)' = \left(\mathcal{B}_{1,1}^{1/2+\kappa}(\mathbb{R})\right)' \subseteq \left(\mathcal{C}_1^{\beta}(\mathbb{R})\right)'$ so that we can define

$$\dot{W}^\varepsilon(t) := \langle \dot{W}, \delta^\varepsilon(t,\cdot)\rangle \mathbf{1}_{\mathbb{R}_+}(t) \tag{7.16}$$

which is a Gaussian process, pathwise measurable and locally bounded. For (maybe stochastic) integrands f we introduce the notation

$$\int_0^t f(r)\,\mathrm{d}W_r^\varepsilon := \int_0^t f(r)\dot{W}_r^\varepsilon\,\mathrm{d}r \tag{7.17}$$

and if f takes values in some (non-homogeneous) Wiener chaos induced by $\dot{W}$ we also introduce the "renormalized" integral

$$\int_0^t f(r) \diamond \mathrm{d}W_r^\varepsilon := \int_0^t f(r) \diamond \dot{W}_r^\varepsilon\,\mathrm{d}r\,, \tag{7.18}$$

where we recall that $\diamond$ denotes the Wick product. Note that the right hand side 7.18 is *defined* by the right hand side, since it can *not* be read as a Skorohod integral as $\dot{W}^\varepsilon$ is no Gaussian stochastic measure in the sense of [Jan97, Definition 7.17]. The integrals (7.17) and (7.18) do in general not coincide, unless f is deterministic. We define an approximate fractional Brownian motion by

$$\hat{W}^\varepsilon(t) = (K * \dot{W}^\varepsilon)(t) = \sqrt{2H}\int_0^t |t-r|^{H-1/2}\,\mathrm{d}W_r^\varepsilon$$

which has the expected regularity as we show in the following lemma.

Lemma 7.1.7. *On every compact time intervall $[0,T]$ we have the estimates*

$$|\hat{W}_t^\varepsilon - \hat{W}_s^\varepsilon| \lesssim K_\varepsilon |t-s|^{H-\kappa'},\ |\hat{W}_t^\varepsilon - \hat{W}_s^\varepsilon - (\hat{W}_t - \hat{W}_s)| \lesssim K^\varepsilon |t-s|^{H-\kappa'}\varepsilon^{\delta\kappa'}\,.$$

uniformly in $\varepsilon \in (0,1]$ for any $\delta \in (0,1)$ and $\kappa' \in (0,H)$ and where $K_\varepsilon > 0$ is a random constant that is (uniformly) bounded in L^p for $p \in [1,\infty)$.

Proof. The proof is elementary but a bit bulky and therefore postponed to Section 7.4 below. □

Finally we can give the definition of the approximative model $(\Pi^\varepsilon, \Gamma^\varepsilon)$, the "canonical" model built from the approximate noise W^ε.

$$
\begin{aligned}
&\Pi_s^\varepsilon \mathbf{1} = 1 && \Gamma_{ts}^\varepsilon \mathbf{1} = 1\\
&\Pi_s^\varepsilon \Xi = \dot{W}^\varepsilon && \Gamma_{ts}^\varepsilon \Xi = \Xi\\
&\Pi_s^\varepsilon \mathcal{I}(\Xi)^m = \left(\hat{W}_\cdot^\varepsilon - \hat{W}_s^\varepsilon\right)^m && \Gamma_{ts}^\varepsilon \mathcal{I}(\Xi) = \mathcal{I}(\Xi) + \left(\hat{W}_t^\varepsilon - \hat{W}_s^\varepsilon\right)\mathbf{1}\\
&\Pi_s^\varepsilon \mathcal{I}(\Xi)^m \Xi = (\hat{W}_\cdot^\varepsilon - \hat{W}_s^\varepsilon)^m \, \dot{W}^\varepsilon && \Gamma_{ts}^\varepsilon \tau\tau' = \Gamma_{ts}^\varepsilon \tau \cdot \Gamma_{ts}^\varepsilon \tau' , \ \tau, \tau', \tau \cdot \tau' \in S
\end{aligned}
$$

Lemma 7.1.8. *The pair $(\Pi^\varepsilon, \Gamma^\varepsilon)$ as defined above is a model (with local bounds) on $(A, \mathcal{T}, G)$.*

Remark 7.1.9. *Concerning local bounds a similar remark as 7.1.4 applies.*

Proof. The identity $\Pi_t^\varepsilon = \Gamma_{ts}^\varepsilon \Pi_s^\varepsilon$ is straightforward to check. The bounds from Definition 2.3.9 on Γ_{st}^ε and on $\Pi_s^\varepsilon \mathcal{I}(\Xi)^m$ follow from the regularity of $\hat{W}^\varepsilon$ as proved in Lemma 7.1.7. The blow-up of $\Pi_s^\varepsilon \Xi \mathcal{I}(\Xi)^m(\varphi_{\cdot - s}^\lambda)$ however is even better than we need, since by the choice of δ^ε we have $|\dot{W}^\varepsilon| \leqslant C_\varepsilon$, for some random constant C_ε (that might diverge for $\varepsilon \to 0$), on compact sets. □

The definition of this model is justified by the fact that one can use the reconstruction operator $\mathcal{R}$ from Theorem 2.3.19 and a suitable defined modelled distribution to build integrals of the type

$$\int_0^t f(\hat{W}_r^\varepsilon, r) \, \mathrm{d}W_r^\varepsilon , \tag{7.19}$$

see Section 7.2 below for details. As pointed out in the introduction, by the Wong-Zakai result [WZ65], there is no hope that integrals of this type will converge as $\varepsilon \to 0$ to anything meaningful for $H < 1/2$. This can be cured by working with a renormalized model $(\hat{\Pi}^\varepsilon, \Gamma^\varepsilon)$ instead.

The renormalized model $\hat{\Pi}^\varepsilon$

From the perspective of regularity structures the fundamental reason why integrals like (7.19) fail to converge to

$$\int_0^t f(\hat{W}_r, r) \, \mathrm{d}W_r$$

lies in the fact that the corresponding models will not satisfy $(\Pi^\varepsilon, \Gamma^\varepsilon) \to (\hat{\Pi}, \hat{\Gamma})$ in the sense of the model distance in Definition 2.3.9, where $(\hat{\Pi}, \hat{\Gamma})$ is the limiting model defined above. To see what is going on we will first rewrite $\Pi_s \Xi \mathcal{I}(\Xi)^k$

Lemma 7.1.10. *For $\varphi \in C_c^\infty(\mathbb{R})$, $s \in \mathbb{R}$, $m \in \{1, \ldots, M\}$ we have for the limiting model $(\hat{\Pi}, \hat{\Gamma})$*

$$\begin{aligned}\hat{\Pi}_s \Xi \mathcal{I}(\Xi)^m(\varphi) &= \int_0^\infty \varphi(t)\,(\hat{W}_t - \hat{W}_s)^m \diamond \mathrm{d}W_t \\ &\quad -m \int_0^\infty \varphi(t)\,K(s-t)\,(\hat{W}_t - \hat{W}_s)^{m-1}\,\mathrm{d}t\end{aligned}$$

where $\diamond$ denotes the Skorohod integral and $K(t) = \sqrt{2H}\mathbf{1}_{t>0}t^{H-1/2}$ denotes the Volterra kernel. Note that in the second term the domain of integration is actually $(0, s)$.

Proof. We prove this by reexpressing $\mathbb{W}_{s,t}^m$. For $s < t$ we have already

$$\mathbb{W}_{s,t}^m = \int_s^t \mathrm{d}W_r \diamond (\hat{W}_r - \hat{W}_s)^m$$

so that it remains to see what happens for $t < s$. With relation (7.15) we have in this case

$$\mathbb{W}_{s,t}^m = -\sum_{l=0}^m \binom{m}{l} (\hat{W}_t - \hat{W}_s)^l \cdot \int_t^s \mathrm{d}r\,\dot{W}_r \diamond (\hat{W}_r - \hat{W}_t)^{m-l},$$

where we use for the sake of concision *formal notation*, which is however easy to translate to a rigorous formulation. Using the fact that for Gaussians U_1, V, U_2 we have

$$U_1^l \cdot (V \diamond U_2^{m-l}) = V \diamond (U_1^l U_2^{m-l}) + l\mathbb{E}[VU_1]\,U_1^{l-1}U_2^{m-l} \tag{7.20}$$

(a consequence of [Jan97, Theorems 3.15, 7.33]), we obtain

$$\begin{aligned}\mathbb{W}_{s,t}^m &= -\int_t^s \mathrm{d}r\,\dot{W}_r \diamond (\hat{W}_r - \hat{W}_s)^m \\ &\quad -\sum_{l=0}^m \binom{m}{l} l \cdot \int_t^s \mathrm{d}r\,\mathbb{E}[\dot{W}_r \cdot (\hat{W}_t - \hat{W}_s)] \cdot (\hat{W}_t - \hat{W}_s)^{l-1} \cdot (\hat{W}_r - \hat{W}_t)^{m-l}\,.\end{aligned}$$

Using $\binom{m}{l} = m\binom{m-1}{l-1}$ and $\mathbb{E}[\dot{W}_r \cdot (\hat{W}_t - \hat{W}_s)] = -K(s-r)$ for $t < r < s$ we can reformulate this and obtain

$$\mathbb{W}_{s,t}^m = -\int_t^s \mathrm{d}W_r \diamond (\hat{W}_r - \hat{W}_s)^m + m\int_t^s \mathrm{d}r K(s-r)(\hat{W}_r - \hat{W}_s)^{m-1}\,.$$

Since $\hat{\Pi}_s \Xi \mathcal{I}(\Xi)^m(\varphi) = \int \varphi(t)\,\mathrm{d}(\mathbb{W}_{s,\cdot}^m)_t$ the claim follows. □

Let us also reexpress the approximating model in a suitable form.

Lemma 7.1.11. *For $\varphi \in C_c^\infty(\mathbb{R})$, $s \in \mathbb{R}$, $m \in \{1, \dots, M\}$ we have*

$$\begin{aligned}\Pi_s^\varepsilon \Xi \mathcal{I}(\Xi)^m(\varphi) = &\int_0^\infty \varphi(t)\,(\hat{W}_t^\varepsilon - \hat{W}_s^\varepsilon)^m \diamond \mathrm{d}W_t^\varepsilon \\ &- m\int_0^\infty \varphi(t)\,\mathscr{K}^\varepsilon(s,t)(\hat{W}_t^\varepsilon - \hat{W}_s^\varepsilon)^{m-1}\mathrm{d}t \\ &+ m\int_0^\infty \varphi(t)\,\mathscr{K}^\varepsilon(t,t)(\hat{W}_t^\varepsilon - \hat{W}_s^\varepsilon)^{m-1}\,\mathrm{d}t\end{aligned}$$

where $\diamond$ is defined as in (7.18) *and where*

$$\mathscr{K}^\varepsilon(u,v) := \mathbb{E}[\hat{W}_u^\varepsilon\,\dot{W}_v^\varepsilon] = \mathbf{1}_{u,v\geqslant 0}\int_0^\infty\int_0^\infty \delta^\varepsilon(v,x_1)\delta^\varepsilon(x_1,x_2)K(u-x_2)\,\mathrm{d}x_1\mathrm{d}x_2\,. \tag{7.21}$$

Proof. Using that for Gaussian random variables V, U we have $VU^m = V \diamond U^m + m\mathbb{E}[VU]U^{m-1}$ (this is (7.20) with $U_2 = 1$) we can rewrite

$$\begin{aligned}\Pi_s^\varepsilon \Xi \mathcal{I}(\Xi)^m(\varphi) = &\int_0^\infty \varphi(t)\,(\hat{W}_t^\varepsilon - \hat{W}_s^\varepsilon)^m \diamond \mathrm{d}W_t^\varepsilon \\ &+ m\int_0^\infty \mathrm{d}t\,\varphi(t)\,\mathbb{E}[\dot{W}_t^\varepsilon\,(\hat{W}_t^\varepsilon - \hat{W}_s^\varepsilon)](\hat{W}_t^\varepsilon - \hat{W}_s^\varepsilon)^{m-1}.\end{aligned}$$

Replacing $\mathbb{E}[\dot{W}_t^\varepsilon\,(\hat{W}_t^\varepsilon - \hat{W}_s^\varepsilon)] = \mathscr{K}^\varepsilon(t,t) - \mathscr{K}^\varepsilon(s,t)$ shows the identity. □

Comparing the expressions in Lemma 7.1.11 and 7.1.10 we see that we morally have to subtract

$$m\int \varphi(t)\,\mathscr{K}^\varepsilon(t,t)(\hat{W}_t^\varepsilon - \hat{W}_s^\varepsilon)^{m-1}\,\mathrm{d}t$$

from the model, which will give us a new model $\hat{\Pi}^\varepsilon$. Of course we have to be careful that this step preserves "Chen's relation" $\hat{\Pi}_s^\varepsilon\Gamma_{st}^\varepsilon = \hat{\Pi}_t^\varepsilon$, see Theorem 7.1.13 below.

If we interpret $\mathscr{K}^\varepsilon$ as an approximation to the Volterra-kernel we see that the expression

$$\mathscr{C}^\varepsilon(t) := \mathscr{K}^\varepsilon(t,t),\ t \geqslant 0$$

will correspond to something like "$0^{H-1/2} = \infty$" in the limit $\varepsilon \to 0$. From this point of view the following upper bound seems quite natural.

Lemma 7.1.12. *For all $s, t \in \mathbb{R}$ we have*

$$|\mathscr{K}^\varepsilon(s,t)| \lesssim \varepsilon^{H-1/2}\,.$$

Proof. $|\mathscr{K}^\varepsilon(s,t)| \lesssim \varepsilon^{-2}\int_{B(t,c\varepsilon)} \mathrm{d}x \int_{B(x,c\varepsilon)} \mathrm{d}u\, |s-u|^{H-1/2} \lesssim \varepsilon^{H-1/2}$. □

Our hope is now that the new model $\hat{\Pi}^\varepsilon$ converges to $\hat{\Pi}$ in the sense of the "model distance" in Definition 2.3.9.

The following, fundamental result of this subsection answers this question and plays a key role in our approximation theory.

Theorem 7.1.13. *Define, for every $s \in [0,T]$, the linear map $\hat{\Pi}^\varepsilon_s : \mathcal{T} \to C^1_c(\mathbb{R})'$ by*

$$\hat{\Pi}^\varepsilon_s \Xi \mathcal{I}(\Xi)^m = \Pi^\varepsilon_s \Xi \mathcal{I}(\Xi)^m - m\mathscr{C}^\varepsilon(\cdot)\Pi^\varepsilon_s(\mathcal{I}(\Xi)^{m-1})$$

for $m \in \{1,\dots,M\}$ and by $\hat{\Pi}^\varepsilon_s = \Pi^\varepsilon_s$ on all remaining symbols in S. Then

$$(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon) := (\hat{\Pi}^\varepsilon, \Gamma^\varepsilon)$$

defines a model (with local bounds) on $(A,\mathcal{T},G)$. On compact time intervals $[0,T]$ we have for $\gamma \in \mathbb{R}$

$$|\!|\!|(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon); (\hat{\Pi}, \hat{\Gamma})|\!|\!|_{\gamma,[0,T]} \lesssim K^\varepsilon \varepsilon^{\delta\kappa}\,. \tag{7.22}$$

for any $\delta \in (0,1)$, κ is as on page 177 and where K^ε is a random constant that is uniformly bounded in L^p for $p \in [1,\infty)$.

In particular, we have "almost rate H" for $M = M(\kappa,H)$ large enough.

Remark 7.1.14. *Concerning the local bounds of $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$ a similar remark as 7.1.4 applies.*

Proof. Since we have, for fixed ε, that $\sup_{t\in[0,T]} |\mathscr{C}^\varepsilon(t)| < \infty$ (Lemma 7.1.12) and $|\Pi^\varepsilon_s \mathcal{I}(\Xi)^m| \lesssim |\cdot - s|^{mH}$ (Lemma 7.1.8) the bound (2.39) is still satisfied. The modification $\hat{\Pi}^\varepsilon_s \Xi\mathcal{I}(\Xi)^k - \Pi^\varepsilon_s \Xi\mathcal{I}(\Xi)^k$ does further not lead to a violation of "Chen's relation"

$$\hat{\Pi}^\varepsilon_t \Gamma^\varepsilon_{ts} \Xi\mathcal{I}(\Xi)^m = \hat{\Pi}^\varepsilon_s \Xi\mathcal{I}(\Xi)^m.$$

Indeed, using validity of (2.37) for $(\Pi^\varepsilon, \Gamma^\varepsilon)$, we have

$$\begin{aligned}
\hat{\Pi}^\varepsilon_t \Gamma^\varepsilon_{ts} \Xi\mathcal{I}(\Xi)^m &= \hat{\Pi}^\varepsilon_t \left(\sum_{l=0}^m \binom{m}{l} (\hat{W}^\varepsilon_t - \hat{W}^\varepsilon_s)^l \Xi \mathcal{I}(\Xi)^{m-l} \right) \\
&= \Pi^\varepsilon_s \Xi\mathcal{I}(\Xi)^m - \sum_{l=0}^m \binom{m}{l} (\hat{W}^\varepsilon_t - \hat{W}^\varepsilon_s)^l (m-l)\mathscr{C}^\varepsilon(\cdot)(\hat{W}^\varepsilon_\cdot - \hat{W}^\varepsilon_t)^{m-l-1} \\
&= \Pi^\varepsilon_s \Xi\mathcal{I}(\Xi)^m - m\mathscr{C}^\varepsilon(\cdot) \sum_{l=0}^{m-1} \binom{m-1}{l} (\hat{W}^\varepsilon_t - \hat{W}^\varepsilon_s)^l \, (\hat{W}^\varepsilon_\cdot - \hat{W}^\varepsilon_t)^{m-l-1} \\
&= \Pi^\varepsilon_s \Xi\mathcal{I}(\Xi)^m - m\mathscr{C}^\varepsilon(\cdot)(\hat{W}^\varepsilon_\cdot - \hat{W}^\varepsilon_s)^m = \hat{\Pi}^\varepsilon_s \Xi(\mathcal{I}(\Xi)^m\,.
\end{aligned}$$

This already shows that $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$ is still a model on $(A, \mathcal{T}, G)$.

We now prove (7.22). We only consider the symbols $\Xi\mathcal{I}(\Xi)^m$, the symbols $\mathcal{I}(\Xi)^m$ can be handled with Lemma 7.1.7. In view of Lemma 7.1.10 and 7.1.11 we have to control for $s \in [0,T]$, $\varphi \in C_c^\infty(B(0,1))$ and $\lambda \in (0,1]$ (with $m \geqslant 0$ in the first equation and $m > 0$ in the second equation)

$$\mathbb{E}\left|\int_0^\infty \mathrm{d}W_t^\varepsilon \diamond \varphi_{t-s}^\lambda (\hat{W}_{st}^\varepsilon)^m - \int_0^\infty \mathrm{d}W_t \diamond \varphi_{t-s}^\lambda (\hat{W}_{st})^m\right|^2 \lesssim \varepsilon^{2\delta\kappa'} \lambda^{2mH-1-2\kappa'}\,, \tag{7.23}$$

$$\mathbb{E}\left|\int_0^\infty \mathrm{d}t\, \varphi_{t-s}^\lambda \left(\mathscr{K}^\varepsilon(s,t)(\hat{W}_{st}^\varepsilon)^{m-1} - K(s-t)(\hat{W}_{st})^{m-1}\right)\right|^2 \lesssim \varepsilon^{2\delta\kappa'} \lambda^{2mH-1-2\kappa'}\,, \tag{7.24}$$

where $\hat{W}_{st}^\varepsilon = \hat{W}_t^\varepsilon - \hat{W}_s^\varepsilon$ and similar for W_{st} and where $\delta \in (0,1)$, $\kappa' \in (0,H)$ is arbitrary. Equivalence of norms in the Wiener chaos and a version of Kolmogorov's criterion for models ([Hai14, Proposition 3.32]) then gives (7.22) (note that this gives for a better homogeneity then we actually need since we only subtract $2\kappa'$ and not $2m\kappa'$ in the exponent of $\lambda \in (0,1]$). We can rewrite the random variable of (7.23) as

$$\int_0^{T+1+c} \mathrm{d}W_t \diamond \int \mathrm{d}u\, \delta^\varepsilon(t,u) \left(\mathbf{1}_{u\geqslant 0}\varphi_{u-s}^\lambda (\hat{W}_{su}^\varepsilon)^m - \varphi_{t-s}^\lambda (\hat{W}_{st})^m\right)\,,$$

where $c > 0$ is as in Definition 7.1.5.

Using [Jan97, Theorem 7.39] and Jensen's inequality we can estimate the second moment of this Skorohod integral by

$$\mathbb{E}|(7.23)|^2 \lesssim \int_0^{T+1+c} \mathrm{d}t \int \mathrm{d}u\, |\delta^\varepsilon(t,u)|\, \mathbb{E}\left(\mathbf{1}_{u\geqslant 0}\varphi_{u-s}^\lambda (\hat{W}_{su}^\varepsilon)^m - \varphi_{t-s}^\lambda (\hat{W}_{st})^m\right)^2\,.$$

In the regime $\lambda \leqslant \varepsilon$ every term in the parentheses can simply be bounded (using Lemma 7.1.7) by $\lambda^{2H-1} \lesssim \lambda^{2H-1-2\kappa'}\varepsilon^{\kappa'}$. If on the other hand $\varepsilon < \lambda$ we can split off a term of order $\int_{B(0,c\varepsilon)} \mathrm{d}t \int_{B(0,c\varepsilon)} \frac{\mathrm{d}u}{\varepsilon} \lesssim \varepsilon \lesssim \lambda^{2mH-1-2\kappa'}\varepsilon^{2\kappa'}$ to drop the indicator $\mathbf{1}_{u\geqslant 0}$ and can bound on the support of $\delta^\varepsilon(t,u)$ via Lemma 7.1.7 and interpolation

$$\begin{aligned}
|\varphi_{u-s}^\lambda (\hat{W}_{su}^\varepsilon)^m - \varphi_{t-s}^\lambda (\hat{W}_{ts})^m| &\leqslant |(\varphi_{u-s}^\lambda - \varphi_{t-s}^\lambda)\cdot|\hat{W}_{su}^\varepsilon|^m + |\varphi_{t-s}^\lambda|\cdot\left|(\hat{W}_{su}^\varepsilon)^m - (\hat{W}_{st})^m\right| \\
&\lesssim K^\varepsilon \mathbf{1}_{B(s,(1+2c)\lambda)}(t)\, \lambda^{-1-\kappa'}\varepsilon^{\kappa'}\lambda^{mH} \\
&\quad + K^\varepsilon \mathbf{1}_{B(s,\lambda)}(t)\lambda^{-1}\lambda^{mH-\kappa'}\varepsilon^{\kappa'}\,,
\end{aligned}$$

where $K^\varepsilon > 0$ denote random constants that are uniformly bounded in L^p for $p \in [1,\infty)$. This shows (7.23). To estimate (7.24) we first note that due to $\mathbb{E}|(\hat{W})_{st}^{m-1} -$

$(\hat{W}^{\varepsilon}_{st})^{m-1}|^2 \lesssim |t-s|^{2(m-1)H-2\kappa'}\varepsilon^{\delta 2\kappa'}$ we are only left with

$$\begin{aligned}&\mathbb{E}\left|\int_0^\infty \mathrm{d}t\,\varphi^\lambda_{t-s}\,(\mathscr{K}^\varepsilon(s,t)-K(s-t))\,(\hat{W}^\varepsilon_{st})^{m-1}\right|^2\\ &\lesssim \int_0^\infty \mathrm{d}t\,\varphi^\lambda_{t-s}\,|\mathscr{K}^\varepsilon(s,t)-K(s-t)|^2\;|s-t|^{2(m-1)H}\,,\end{aligned}$$

which is straightforward to bound with Lemma 7.1.12 if $\lambda \leqslant \varepsilon$. For $\lambda > \varepsilon$ and $t > 2c\varepsilon$ with $c > 0$ as in Definition 7.1.5 the desired bound follows from Lemma 7.4.3 below. The remaining case however contributes with

$$\begin{aligned}&\int_{B(0,2c\varepsilon)} \mathrm{d}t\,\varphi^\lambda_{t-s}|t-s|^{2(m-1)H}(\varepsilon^{2H-1}+|t-s|^{2H-1})\\ &\lesssim \int_{B(s,\lambda^{-1}2c\varepsilon)} \mathrm{d}t\,(\lambda^{2(m-1)H}\varepsilon^{2H-1}+\lambda^{2mH-1}|t|^{2mH-1})\\ &\lesssim \lambda^{2(m-1)H-1}\varepsilon^{2H}+\lambda^{2mH-1}(\lambda^{-1}\varepsilon)^{2mH} \lesssim \lambda^{2mH-\kappa'}\varepsilon^{\kappa'}\,,\end{aligned}$$

which completes the proof. □

7.2 Approximation theory via reconstruction

We now address the central question of this chapter of how the integral

$$\int_0^t f(\hat{W}^\varepsilon(r),r)\,\mathrm{d}W^\varepsilon(r)$$

has to be modified to make it convergent against $\int_0^t f(W(r),r)\mathrm{d}W(r)$. To this end we describe the integrand $f(\hat{W}^{(\varepsilon)}_r,r)$ by a function $F:[0,T]\to\mathcal{T}$

$$F^{(\varepsilon)}(s) := \sum_{m=0}^{M}\frac{1}{m!}\partial_1^m f(\hat{W}^{(\varepsilon)}_s,s)\mathcal{I}(\Xi)^m \tag{7.25}$$

We here used the representation "$X^{(\varepsilon)}$" to denote both X and X^ε. We will occasionally use this shorthand notation in this section to shorten the formulas a bit. The notation ∂_1^m in (7.25) should be read for $m \geqslant 0$ as

$$\partial_1^m := \partial^{(m,0)}$$

and will further write ∂_1 whenever $m=1$. Once we have shown (7.25) is a modelled distribution in $\mathcal{D}^\gamma([0,T];\mathcal{T},\hat{\Gamma}^{(\varepsilon)})$ with $\gamma>0$ large enough we can multiply it with the symbol Ξ and then apply the reconstruction operator $\mathcal{R}$ from Theorem 2.3.19. The easiest way to show that $F^{(\varepsilon)}$ is a modelled distribution is to read it as in Lemma 5.3.9 as the composition of the smooth function f with the pair of modelled distributions $(\mathcal{K}^{(\varepsilon)}\Xi, \mathbf{1})$ where

$$(\mathcal{K}^{(\varepsilon)}\Xi)(t) := \hat{W}_t^{(\varepsilon)}\,\mathbf{1} + \mathcal{I}(\Xi)$$

A similar notation for the "operator" $\mathcal{K}$ as in Section 6.2 and as in [Hai14, Section 5] was chosen on purpose, since $\mathcal{K}^{(\varepsilon)}\Xi$ can be seen as the integration of $\Xi \in \mathcal{D}^\infty(\mathbb{R}^d; \mathcal{T}, \hat{\Gamma}^{(\varepsilon)}) = \bigcap_{\gamma\in\mathbb{R}} \mathcal{D}^\gamma(\mathbb{R}^d; \hat{\Gamma}^{(\varepsilon)})$ against the Volterra-Kernel K. One easily verifies by direct computation that $\mathbf{1}$, $\mathcal{K}^{(\varepsilon)}\Xi \in \mathcal{D}^\infty(\mathbb{R}^d; \mathcal{T}, \hat{\Gamma}^{(\varepsilon)})$. This allows us to apply [Hai14, Thm 4.16], which is the non-singular analogue of Lemma 5.3.9, to obtain the following estimate.

Lemma 7.2.1. *Given $f \in C_b^{2M+3}(\mathbb{R} \times [0,T])$ one has for all $\gamma \in (1/2+\kappa, 1)$,*

$$\|F^{(\varepsilon)}\|_{\mathcal{D}^\gamma([0,T];\mathcal{T},\hat{\Gamma}^{(\varepsilon)})} + \|\Xi F^{(\varepsilon)}\|_{\mathcal{D}^{\gamma+|\Xi|}([0,T];\mathcal{T},\hat{\Gamma})} \lesssim K_1^{(\varepsilon)}\,,$$

where $K_1^{(\varepsilon)}$ is a polynomial (with coefficients independent of ε) in $\|(\hat{\Pi}^{(\varepsilon)}, \hat{\Gamma}^{(\varepsilon)})\|_\gamma$. We have further

$$\|F^\varepsilon; F\|_{\mathcal{D}^\gamma([0,T];\mathcal{T};\hat{\Gamma}^\varepsilon,\hat{\Gamma})} + \|\Xi F^\varepsilon; \Xi F\|_{\mathcal{D}^{\gamma+|\Xi|}([0,T];\mathcal{T};\hat{\Gamma}^\varepsilon,\hat{\Gamma})} \lesssim K_2 \cdot |\!|\!|(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon); (\hat{\Pi}, \hat{\Gamma})|\!|\!|_\gamma \quad (7.26)$$

K_2^ε is a polynomial (with coefficients independent of ε) in the corresponding norms of F, F^ε, $(\hat{\Pi}, \hat{\Gamma})$ and $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$.

Proof. The estimates on $\|F^{(\varepsilon)}\|_{\mathcal{D}^\gamma([0,T];\Gamma^{(\varepsilon)})}$ and $\|F^\varepsilon; F\|_{\mathcal{D}^\gamma([0,T];\mathcal{T};\hat{\Gamma}^\varepsilon,\hat{\Gamma})}$ follow with $\mathbf{1}$, $\mathcal{K}^{(\varepsilon)}\Xi \in \mathcal{D}^\infty(\mathbb{R}^d; \Gamma^{(\varepsilon)})$ from [Hai14, Thm 4.16] (polynomial dependence on the model norm is not stated there but is clear from the proof). The estimates involving $\Xi F^{(\varepsilon)}$ then follow by an application of [Hai14, Thm 4.7, Prop. 4.10]. $\square$

Remark 7.2.2. *In the case when $f \in C^{2M+3}$ but with no global bounds, the result still holds since we only consider the values of f on the range of the continuous function $\mathcal{R}\mathcal{K}\Xi$ (which is bounded by some $R \geqslant 0$). The resulting bounds then depend linearly on $\|f\|_{C^{2M+3}(B_R\times[0,T])}$.*

We are thus allowed to apply the reconstruction operator from Theorem 2.3.19 to $F^{(\varepsilon)}$, which we expect to yield a distribution that somehow describes $f(\hat{W}^{(\varepsilon)}, t)\,\dot{W}^{(\varepsilon)}$ in a way that is robust under approximations.

Lemma 7.2.3. *We have (a.s.)*

$$\begin{aligned}
\mathcal{R}F\Xi(\varphi) &= \int \varphi(t)\, f(\hat{W}(t), t)\, \mathrm{d}W_t\,, \\
\mathcal{R}^\varepsilon F^\varepsilon\Xi(\varphi) &= \int \varphi(t)\, f(\hat{W}_t^\varepsilon, t)\, \mathrm{d}W_t^\varepsilon - \int \mathscr{K}^\varepsilon(t,t)\partial_1 f(\hat{W}_t^\varepsilon, t)\varphi(t)\, \mathrm{d}t\,.
\end{aligned}$$

where $\mathcal{R}^{(\varepsilon)}$ is the reconstruction operator from Theorem 2.3.19 for the model $(\hat{\Pi}^{(\varepsilon)}, \hat{\Gamma}^{(\varepsilon)})$.

Proof. For $\varphi \in C_c^\infty(\mathbb{R})$ and $s \in \mathbb{R}$ we have

$$\begin{aligned}
&\mathbb{E}\left|\int \varphi^\lambda_{t-s} f(\hat{W}_t, t)\,\mathrm{d}W_t - \hat{\Pi}_s F_s \Xi(\varphi^\lambda_{\cdot-s})\right|^2 \\
&= \mathbb{E}\left|\int \varphi^\lambda_{t-s}(f(\hat{W}_t, t) - \sum_{m=0}^{M} \partial_1^m f(\hat{W}_s, s)(\hat{W}_t - \hat{W}_s)^m)\mathrm{d}W_t\right|^2 \\
&= \int_{\mathbb{R}_+} \mathbb{E}\left|\varphi^\lambda_{t-s}(f(\hat{W}_t, t) - \sum_{m=0}^{M} \partial_1^m f(\hat{W}_s, s)(\hat{W}_t - \hat{W}_s))\right|^2 \mathrm{d}t \lesssim_\varphi \lambda^{2\gamma}
\end{aligned}$$

for $0 < \gamma < [(M+1)(H-\kappa) \wedge 1] - 1/2$ where we applied Taylor's formula and Lemma 7.1.7. Proceeding now as in the proof of [Hai14, Theorem 3.10] we choose test functions $\eta, \psi \in C_c^\infty$ with η even and $\operatorname{supp}\eta \subseteq \mathrm{B}(0,1)$, $\int \eta(\mathrm{t})\,\mathrm{dt} = 1$. We then obtain for $\psi^\delta = \psi * \eta^\delta$

$$\begin{aligned}
&\mathbb{E}\left[\left|\mathcal{R}F\Xi(\psi^\delta) - \int \psi^\delta(t)\, f(\hat{W}_t, t))\mathrm{d}W_t\right|^2\right] \\
&= \mathbb{E}\Big[\Big| \int \mathrm{d}x\, \psi(x) \Big(\mathcal{R}F\Xi(\eta^\delta_{\cdot-x}) - \hat{\Pi}_x \Xi F(\eta^\delta_{\cdot-x}) + \hat{\Pi}_x \Xi F(\eta^\delta_{\cdot-x}) \\
&- \int \eta^\delta_{t-x}\, f(\hat{W}_t, t))\mathrm{d}W_t\Big)\Big|^2\Big] \\
&\lesssim \int \mathrm{d}x\, \psi^2(x)\, \delta^{2\gamma} \overset{\delta\to 0}{\to} 0
\end{aligned}$$

where we included a term $\hat{\Pi}_x \Xi F(\eta^\delta_{\cdot-x})$ in the second step and then used Jensen's inequality. It remains to note that

$$\int \psi^\delta(t)\, f(\hat{W}_t, t))\mathrm{d}W_t \overset{\delta\to 0}{\to} \int \psi(t)\, f(\hat{W}_t, t)\mathrm{d}W_t$$

in $L^2(\mathbb{P})$ and further $\mathcal{R}F\Xi(\psi^\delta) \to \mathcal{R}F\Xi(\psi)$ a.s. and thus in $L^2(\mathbb{P})$. Putting everything together we obtain

$$\mathbb{E}\left[|\mathcal{R}F\Xi(\psi) - \int \psi(t)\, f(\hat{W}_t, t)\mathrm{d}W_t|^2\right] = 0$$

which implies the first statement. For the second identity we proceed in the same way but make use of Lemma 7.4.4 below. □

Integration of this expression then corresponds to testing this distribution against $\mathbf{1}_{[0,T]}$. This is a well-defined operation since the reconstruction lies by Theorem 2.3.19 (locally) in $\mathcal{C}^{-1/2-\kappa}(\mathbb{R})$, which is included by Lemma 2.1.29 and 2.1.27 in

$$\mathcal{C}^{-1/2-\kappa}(\mathbb{R}) \subseteq (\mathcal{C}_1^1(\mathbb{R}))' .$$

Since $\mathbf{1}_{[0,T]} \in \mathcal{C}_1^1(\mathbb{R})$ the integration of $\mathcal{R}^{(\varepsilon)}\Xi F^\varepsilon = f(\hat{W}^{(\varepsilon)}, t)\,\dot{W}^{(\varepsilon)}$ over $[0,T]$ can then be read as a dual pairing. We will need that the bounds from Theorem 2.3.19 and Definition 2.3.9 are well-behaved with such pairings. This is the content of the following Lemma, in contrast to [BFG$^+$17] we here give a proof which is more coherent with the methods presented in Section 2.1 and Chapter 5.

Lemma 7.2.4. *Let $\mathscr{T} = (A, \mathcal{T}, G)$ be a regularity structure on $\mathbb{R}^d$ with a model (Π, Γ) and let $\varphi \in \mathcal{C}^\beta_{1,\mathfrak{s}}(\mathbb{R}^d)$ with $\beta > -\min A$, $\operatorname{supp}\varphi \subseteq B_\mathfrak{s}(0,1)$ and $\mathfrak{s}$ being the scaling of the regularity structure $\mathcal{T}$. For $x \in \mathbb{R}^d$ it then holds for $\lambda \in (0,1]$*

$$|\Pi_x \tau(\varphi^\lambda_{\cdot - x})| \lesssim K_1 \cdot \lambda^\alpha \|\varphi\|_{\mathcal{C}^\beta_{1,\mathfrak{s}}(\mathbb{R}^d)} , \tag{7.27}$$

$$|(\mathcal{R}F - \Pi_x F_x)(\varphi^\lambda_{\cdot - x})| \lesssim K_2 \cdot \lambda^\gamma \|\varphi\|_{\mathcal{C}^\beta_{1,\mathfrak{s}}(\mathbb{R}^d)} , \tag{7.28}$$

where $\tau \in \mathcal{T}_\alpha$ with $\alpha \in A$, $F \in \mathcal{D}^\gamma(\mathbb{R}^d; \mathcal{T})$ with $\gamma > 0$, $\mathcal{R}$ is the reconstruction operator from Theorem 2.3.19 and K_1, K_2 are some polynomials in $\|\!|(\Pi,\Gamma)|\!\|_{\alpha+1}$ and $\|\!|(\Pi,\Gamma)|\!\|_\gamma$ respectively. Given a second model $(\tilde{\Pi}, \tilde{\Gamma})$ we further have

$$|(\Pi_x \tau - \tilde{\Pi}_x \tau)(\varphi^\lambda_{\cdot - x})| \lesssim K_1 \cdot \|\!|(\Pi,\Gamma); (\tilde{\Pi},\tilde{\Gamma})|\!\|_{\alpha+1}\, \lambda^\alpha \|\varphi\|_{\mathcal{C}^\beta_{1,\mathfrak{s}}(\mathbb{R}^d)} , \tag{7.29}$$

$$\begin{aligned} |(\mathcal{R}F - \Pi_x F_x - \tilde{\mathcal{R}}\tilde{F} - \tilde{\Pi}_x \tilde{F}_x)(\varphi^\lambda_{\cdot - x})| &\lesssim K_2 \\ \times \big(\|\!|(\Pi,\Gamma); (\tilde{\Pi},\tilde{\Gamma})|\!\|_\gamma + \|F; \tilde{F}\|_{\mathcal{D}^\gamma(\mathbb{R}^d; \mathcal{T}, \Gamma, \tilde{\Gamma})}\big)\, &\lambda^\gamma \|\varphi\|_{\mathcal{C}^\beta_{1,\mathfrak{s}}(\mathbb{R}^d)} , \end{aligned} \tag{7.30}$$

where $\tilde{F} \in \mathcal{D}^\gamma(\mathbb{R}^d; \tilde{\Gamma})$ and $\tilde{\mathcal{R}}$ denotes the reconstruction operator for the model $(\tilde{\Pi}, \tilde{\Gamma})$. The polynomials K_1, K_2 are now taken in the corresponding norms of (Π, Γ), $(\tilde{\Pi}, \tilde{\Gamma})$ and (Π, Γ), $(\tilde{\Pi}, \tilde{\Gamma})$, F, $\tilde{F}$ respectively.

Remark 7.2.5. *In the framework in this chapter, with $\beta \in (\frac{1}{2} + \kappa, 1]$, this covers in particular functions like $\varphi = \mathbf{1}_{[0,1/2]} \in \mathcal{C}_1^1(\mathbb{R})$ (Lemma 7.4.1 below) and thus also the Haar basis.*

Remark 7.2.6. *The reader might be worried since we work here with a model with global bounds as in Definition 2.3.9, while our models defined so far in this chapter are all models with local bounds as on page 47. This is however no real restriction since all the estimates above involve compactly supported function so that we can apply these bounds to the models $(\hat{\Pi}, \hat{\Gamma})$, $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$, $(\Pi^\varepsilon, \Gamma^\varepsilon)$ by proceeding as in 7.1.4.*

A similar remark applies to the modelled distribution $F^{(\varepsilon)}$ for which we can apply the Whitney extension result of Theorem 5.3.16.

For a result that does not require global, but only local bounds compare [BFG$^+$17], where the result above is proved with wavelet techniques instead.

Proof. In [BFG+17] this was proved via wavelet methods. We here choose to give a distinct proof which is nevertheless similar in spirit but relies on the Fourier methods we presented in Chapter 2. We only show (7.28), since the remaining estimates (7.27), (7.29) and (7.30) follow by basically the same arguments. By Lemma A.4 of [GIP15] it follows that for $\sigma \in [c, C]$ with $0 < c < C$ one has

$$\|\varphi(\sigma\cdot)\|_{\mathcal{C}^{\beta}_{1,\mathfrak{s}}(\mathbb{R}^d)} \lesssim_{c,C} \|\varphi\|_{\mathcal{C}^{\beta,\mathfrak{s}}_{1}(\mathbb{R}^d)}\,.$$

([GIP15, Lemma A.4] is actually for the isotropic case and $\mathcal{C}^{\beta}_{\infty}$ instead, the translation to the case used here however is rather straightforward). We can thus take dyadic $\lambda = 2^{-N}$ in (7.28) for $N \geqslant 0$. Since $\beta > 0$ by assumption we have due to Lemma 2.1.28 the estimate $\|\varphi\|_{L^1(\mathbb{R}^d)} \lesssim \|\varphi\|_{\mathcal{C}^{\beta}_{1,\mathfrak{s}}(\mathbb{R}^d)}$, which we will use throughtout the proof without further mentioning. Without loss of generality we pick $\|\varphi\|_{\mathcal{C}^{\beta}_{1,\mathfrak{s}}(\mathbb{R}^d)} \leqslant 1$.

Using Littlewood-Paley-decomposition of $\mathcal{R}F - \Pi_x F_x$ we can split the right hand side of (7.28) into

$$\begin{aligned}(\mathcal{R}F - \Pi_x F_x)(\varphi^{\lambda}_{\cdot - x}) &= \int \mathrm{d}y\,(\mathcal{R}F - \Pi_x F_x)(\Psi^{<N}_{\cdot - y}) \cdot \varphi^{\lambda}_{y-x} \\ &+ \sum_{j \geqslant N} \int \mathrm{d}y\,(\mathcal{R}F - \Pi_x F_x)(\Psi^{j}_{\cdot - y}) \cdot \varphi^{\lambda}_{y-x} \end{aligned} \tag{7.31}$$

We split the first term of (7.31) into

$$\int \mathrm{d}y\,(\mathcal{R}F - \Pi_y F_y)(\Psi^{<N}_{\cdot - y}) \cdot \varphi^{\lambda}_{y-x} + \sum_{\alpha < \gamma} \int \mathrm{d}y\, \Pi_y(F^{\alpha}_y - \Gamma^{\alpha}_{yx} F_x)(\Psi^{<N}_{\cdot - y}) \cdot \varphi^{\lambda}_{y-x} \tag{7.32}$$

It was shown in [GIP15, Lemma 6.6] and Lemma 2.3.11 that

$$(\mathcal{R}F - \Pi_y F_y)(\Psi^{<N}_{\cdot - y}) \lesssim K_2 \cdot 2^{-N\gamma},\ \Pi_y(\Psi^{<N}_{\cdot - y}) \lesssim K_2 \cdot 2^{-N\alpha} \tag{7.33}$$

where $\tau \in \mathcal{T}_{\alpha}$ with $\alpha \in A$. Similar bounds hold for Ψ^N as one sees by writing $\Psi^N = \Psi^{<N+1} - \Psi^{<N}$. From this one gets, using $F \in \mathcal{D}^{\gamma}(\mathbb{R}^d; \mathcal{T})$, that (7.32) is bounded by $K_2\, 2^{-N\gamma} = K_2\, \lambda^{\gamma}$. The second term of (7.31) is a bit more delicate, we use spectral support properties to introduce a Littlewood-Paley block $\overline{\Delta}_j = \sum_{j' : |j' - j| \leqslant 1} \Delta_{j'}$ (as in the proof of Lemma 2.1.23) and then perform a splitting similar as for (7.32):

$$\sum_{j \geqslant N} \int \mathrm{d}y\,(\mathcal{R}F - \Pi_x F_x)(\Psi^{j}_{\cdot - y})\,(\overline{\Delta}_j \varphi^{\lambda})_{y-x} = \sum_{j \geqslant N} \int \mathrm{d}y\,(\mathcal{R}F - \Pi_y F_y)(\Psi^{j}_{\cdot - y})\,(\overline{\Delta}_j \varphi^{\lambda})_{y-x} \tag{7.34}$$

$$+ \sum_{j \geqslant N} \sum_{\alpha < \gamma} \int \mathrm{d}y\, \Pi_y(F^{\alpha}_y - \Gamma^{\alpha}_{yx} F_x)(\Psi^{j}_{\cdot - y})\,(\overline{\Delta}_j \varphi^{\lambda})_{y-x} \tag{7.35}$$

Using $\|\overline{\varDelta}_j\varphi^\lambda\|_{L^1(\mathbb{R}^d)} \lesssim \|\varphi^\lambda\|_{L^1(\mathbb{R}^d)} \lesssim 1$ (from (2.20)) the term (7.34) is again, due to (7.33) for $\Psi^j = \Psi^{<j+1} - \Psi^{<j}$, bounded by $\sum_{j\geqslant N} K_2\, 2^{-j\gamma} \lesssim K_2\,\lambda^\gamma$, so that we are left with (7.35) for which we finally apply that $\varphi \in \mathcal{C}_1^\beta(\mathbb{R}^d)$, $\operatorname{supp}\varphi \subseteq B_{\mathfrak{s}}(0,1)$. The term (7.35) can be bounded by

$$
\begin{aligned}
&K_2 \sum_{\alpha<\gamma}\sum_{j\geqslant N} 2^{-j\alpha}\int \mathrm{d}y \|y-x\|_{\mathfrak{s}}^{\gamma-\alpha}\,|(\overline{\varDelta}_j\varphi^\lambda)_{y-x}| \\
&= K_2 \sum_{\alpha<\gamma}\sum_{j\geqslant N} 2^{-j\alpha}\int_{\|y-x\|_{\mathfrak{s}}\leqslant 2\lambda} \mathrm{d}y \,\|y-x\|_{\mathfrak{s}}^{\gamma-\alpha}\,|(\overline{\varDelta}_j\varphi^\lambda)_{y-x}| \qquad (7.36)\\
&+K_2 \sum_{\alpha<\gamma}\sum_{j\geqslant N} 2^{-j\alpha}\int_{\|y-x\|_{\mathfrak{s}}> 2\lambda} \mathrm{d}y \,\|y-x\|_{\mathfrak{s}}^{\gamma-\alpha}\,|(\overline{\varDelta}_j\varphi^\lambda)_{y-x}|\,. \qquad (7.37)
\end{aligned}
$$

For (7.36) we obtain the upper bound

$$
\begin{aligned}
&K_2\sum_{\alpha<\gamma}\lambda^{\gamma-\alpha}\sum_{j\geqslant N}2^{-j\alpha}\int_{\|y-x\|_{\mathfrak{s}}\leqslant 2\lambda}\mathrm{d}y\,|(\overline{\varDelta}_j\varphi^\lambda)_{y-x}| \leqslant K_2\sum_{\alpha<\gamma}\lambda^{\gamma-\alpha}\sum_{j\geqslant N}2^{-j\alpha}\int \mathrm{d}y|(\overline{\varDelta}_j\varphi^\lambda)_{y-x}|\\
&= K_2\sum_{\alpha<\gamma}\lambda^{\gamma-\alpha}\sum_{j\geqslant N}2^{-j\alpha}\int \mathrm{d}y|(\overline{\varDelta}_{j-N}\varphi)^\lambda_{y-x}|\\
&= K_2\sum_{\alpha<\gamma}\lambda^{\gamma-\alpha}\sum_{j\geqslant N}2^{-j\alpha}\|(\overline{\varDelta}_{j-N}\varphi)^\lambda\|_{L^1(\mathbb{R}^d)}\\
&\lesssim K_2\sum_{\alpha<\gamma}\lambda^{\gamma-\alpha}\sum_{j\geqslant N}2^{-j\alpha}2^{-(j-N)\beta} \lesssim K_2\sum_{\alpha<\gamma}\lambda^{\gamma-\alpha}\lambda^{-\beta}\sum_{j\geqslant N}2^{-j(\beta+\alpha)} \lesssim K_2\lambda^\gamma\,,
\end{aligned}
$$

where we applied $\lambda = 2^{-N}$ in the second line, which yields by substitution the identity $\overline{\varDelta}_j\varphi^\lambda = (\overline{\varDelta}_{j-N}\varphi)^\lambda$. By substituting $z = 2^{N\mathfrak{s}}(y-x)$ we can reshape (7.37) as

$$
K_2\sum_{\alpha<\gamma}\sum_{j\geqslant N}2^{-j\alpha}\lambda^{\gamma-\alpha}\int_{\|z\|_{\mathfrak{s}}>2}\mathrm{d}y\,\|z\|_{\mathfrak{s}}^{\gamma-\alpha}\,2^{-N|\mathfrak{s}|}|(\overline{\varDelta}_{j-N}\varphi)^\lambda_{2^{-N\mathfrak{s}}z}|\,. \qquad (7.38)
$$

Now, using that $\operatorname{supp}\varphi \in B_{\mathfrak{s}}(0,1)$ and $\|z\|_{\mathfrak{s}} > 2$ we can estimate for arbitrarily large $a > 0$

$$
\begin{aligned}
2^{-N|\mathfrak{s}|}|(\overline{\varDelta}_{j-N}\varphi)^\lambda_{2^{-N\mathfrak{s}}z}| = |(\overline{\varDelta}_{j-N}\varphi)_z| &= 2^{(j-N)|\mathfrak{s}|}\Big|\int \mathrm{d}u\,\Psi(2^{(j-N)\mathfrak{s}}(z-u))\varphi(u)\Big|\\
&\lesssim 2^{-(j-N)(a-1)}\frac{1}{(\|z\|_{\mathfrak{s}}-1)^a}\,.
\end{aligned}
$$

Choosing now a large enough such that $a+\gamma-\min A > |\mathfrak{s}|$ and $a-1+\min A > 0$ one gets for (7.38) the upper bound

$$
K_2\sum_{\alpha<\gamma}\sum_{j\geqslant N}2^{-j\alpha}\lambda^{\gamma-\alpha}\,2^{-(j-N)(a-1)} \lesssim K_s\lambda^\gamma\,,
$$

which closes the proof. □

Having established this technical result we can in principle always use indicator functions in our framework with the same "legality" than smooth functions. If we take $\varphi = \mathbf{1}_{[0,T)}$ we obtain $\mathcal{R}F\Xi(\mathbf{1}_{[0,T)}) = \int_0^T f(\hat{W}_t, t)\,\mathrm{d}W_t$, so that it is natural to define $\mathscr{I}_f^\varepsilon(T) := \mathcal{R}^\varepsilon \Xi F^\varepsilon(\mathbf{1}_{[0,T)})$ as an approximation (we have of course $\mathbf{1}_{[0,T)} = \mathbf{1}_{[0,T]}$ in the sense of distributions, but the usage of $\mathbf{1}_{[0,T)}$ leads to aesthetically more pleasing expressions below). However, note that the key property of the reconstruction operator $\mathcal{R}^{(\varepsilon)}$ is that it is locally close to the corresponding model $\Pi^{(\varepsilon)}$ so that we have a second natural approximation $I^\varepsilon_{f,M}(t)$

Definition 7.2.7. *For f, F^ε as in Lemma 7.2.1 and $t \geqslant 0$ we set*

$$\mathscr{I}_f^\varepsilon(t) := \mathcal{R}^\varepsilon \Xi F^\varepsilon(\mathbf{1}_{[0,t]}) = \int_0^t f(\hat{W}_r^\varepsilon, r)\,\mathrm{d}W_r^\varepsilon - \int_0^t \mathscr{C}^\varepsilon(r)\partial_1 f(\hat{W}_r^\varepsilon, r)\,\mathrm{d}r\,. \tag{7.39}$$

For a (fixed) partition $\{[t_l^\varepsilon, t_{l+1}^\varepsilon)\}$ of $[0,t)$ with $\left|t_{l+1}^\varepsilon - t_l^\varepsilon\right| \lesssim \varepsilon$ we further set

$$\begin{aligned} I_{f,M}^\varepsilon(t) &= \sum_{[t_l^\varepsilon, t_{l+1}^\varepsilon)} \hat{\Pi}_{t_l}^\varepsilon \Xi F_{t_l}^\varepsilon(\mathbf{1}_{[t_l^\varepsilon, t_{l+1}^\varepsilon)}) \\ &= \sum_{[t_l^\varepsilon, t_{l+1}^\varepsilon)} \sum_{m=0}^M \frac{1}{m!}\partial_1^m f(\hat{W}_{t_l^\varepsilon}^\varepsilon, t_l^\varepsilon) \int_{t_l^\varepsilon}^{t_{l+1}^\varepsilon} \left(\hat{W}_r^\varepsilon - \hat{W}_{t_l^\varepsilon}^\varepsilon\right)^m \mathrm{d}W_r^\varepsilon - \\ &- \sum_{m=1}^M \frac{1}{(m-1)!}\partial_1^m f(\hat{W}_{t_l^\varepsilon}^\varepsilon, t_l^\varepsilon) \int_{t_l^\varepsilon}^{t_{l+1}^\varepsilon} \mathscr{C}^\varepsilon(r) \left(\hat{W}_r^\varepsilon - \hat{W}_{t_l^\varepsilon}^\varepsilon\right)^{m-1} \mathrm{d}r\,. \end{aligned}$$

Remark 7.2.8. *To be precise, Lemma 7.2.3 only states that the second identity in* (7.39) *holds for all $t \geqslant 0$ almost surely. However, using Kolmogorov's criterion and integration results for distributions (for example [GIP15, Lemma A.10]) one sees that one can choose a common version for the reconstruction and the right hand side of* (7.39) *so that we have almost surely an identity for all $t \geqslant 0$. If $\dot{W}^\varepsilon$ is almost surely Hölder continuous, one can alternatively use [Hai14, Remark 3.15].*

The following theorem, which can be seen as the fundamental theorem of our regularity structure approach to rough pricing shows that these approximations do both converge.

Theorem 7.2.9. *Fix $T > 0$. For $f \in C_b^\infty(\mathbb{R})$ and $\mathscr{I}_f^\varepsilon$, $I_{f,M}^\varepsilon$ as in Definition 7.2.7 we have*

(i) for any $\delta \in (0,1)$ one has

$$\sup_{t\in[0,T]} \left|\mathscr{I}_f^\varepsilon(t) - \int_0^t f(\hat{W}_r, r)\mathrm{d}W_r\right| \leqslant K^\varepsilon \varepsilon^{\delta H}\,, \tag{7.40}$$

(ii) for every $\delta \in (0,1)$ we can pick $M = M(\delta, H)$ large enough, such that

$$\sup_{t\in[0,T]} \left| I^{\varepsilon}_{f,M}(t) - \int_0^t f(\hat{W}_r, r)\mathrm{d}W_r \right| \leqslant K^{\varepsilon}\varepsilon^{\delta H}\,. \tag{7.41}$$

where in both cases K^{ε} is a random constant (depending on δ) which can be bounded independent of ε in L^p for $p \in [1, \infty)$.

Remark 7.2.10. *By the Itô isometry and Kolmogorov's continuity criterion one finds that there is $\mathcal{C}^{1/2-\kappa}([0,T])$ version of the process $\left(\int_0^t f(\hat{W}_t, t)\,\mathrm{d}W_t\right)_{t\in[0,T]}$, this is the object for which the statement in (i.) and (ii.) is formulated*

Remark 7.2.11. *With regard to (i): although $\mathscr{I}^{\varepsilon}_f(t)$ does not depend on any choice of M, and nor does its limit, the choice of M affects the entire regularity structure and so, implicitly also the reconstruction operator $\mathcal{R}^{\varepsilon}$ used in the definition of $\mathscr{I}^{\varepsilon}_f$, as well as the modelled distribution F^{ε}. The latter, in turn, requires $f \in C^M$ for the construction to make sense. If δ is chosen arbitrarily close to one, f needs to have derivatives of arbitrary order, hence our smoothness assumption.*

Remark 7.2.12 (f of exponential form)**.** *By an easy localization argument one shows that for smooth $f \in C^{\infty}(\mathbb{R})$ (but without any further bounds) ones still has*

$$\sup_{\varepsilon\in(0,1]} \mathbb{P}\left(\sup_{t\in[0,T]} \left| \mathscr{I}^{\varepsilon}_f(t) - \int_0^t f(\hat{W}_r, r)\mathrm{d}W_r \right| \leqslant C\varepsilon^{\delta H} \right) \to 0$$

with $C \to \infty$ (and similar for $I^{\varepsilon}_{f,M}$). The rough Bergomi model from [BFG16], which we sketched at the beginning of this chapter, proposes that f should be of exponential form. Now, the result with L^p-estimates still holds since we only consider the values of f on the range of the continuous function $\hat{W}^{(\varepsilon)}$ (which is bounded, independent of ε, by some random $R \geqslant 0$). As pointed out in Remark 7.2.2, the bounds then depend linearly on $\|f\|_{C^{M+2}(B_R\times[0,T])}$. Since, for us, $\hat{W}^{(\varepsilon)}$ is always a Gaussian process we have (by Fernique) Gaussian concentration for $\sup_{t\in[0,T]} |\hat{W}^{(\varepsilon)}_t|$. So, for instance if f and its derivatives have exponential growth we do have the L^p bounds of the above theorem, for all $p < \infty$.

Proof. Without loss of generality $T \leqslant 1$, otherwise split $[0,T]$ in subintervals. Let us show (7.40) by rewriting

$$\begin{aligned}
&\mathscr{I}^{\varepsilon}_f(t) - \int_0^t f(\hat{W}_r, r)\mathrm{d}W_r = (\mathcal{R}^{\varepsilon}(F^{\varepsilon}\Xi) - \mathcal{R}(F\Xi))(\mathbf{1}_{[0,t]}) \\
&= t\left(\hat{\Pi}^{\varepsilon}_0 \Xi F^{\varepsilon}(0) - \Pi_0 \Xi F(0)\right)\right)(t^{-1}\mathbf{1}_{[0,t]}) \\
&+ t\left(\mathcal{R}^{\varepsilon}\Xi F^{\varepsilon} - \hat{\Pi}^{\varepsilon}_0 \Xi F^{\varepsilon}(0) - (\mathcal{R}\Xi F - \Pi_0 \Xi F(0))\right)(t^{-1}\mathbf{1}_{[0,t)})\,.
\end{aligned}$$

We then obtain the rate $\varepsilon^{\delta\kappa}, \delta \in (0,1)$ using Theorem 7.1.13, Lemma 7.2.1 and Lemma 7.2.4. Letting $\kappa \uparrow H$ and $M \uparrow \infty$ our total rate can be chosen arbitrarily close to H.

To obtain the second estimate we can bound $\mathscr{I}^\varepsilon_f(t) - I^\varepsilon_{f,M}(t)$ once more with Lemma 7.2.4. □

Non-constant vs. constant renormalization

If δ^ε comes from a mollifier (cf. Example 7.1.6) the renormalization $\mathscr{C}^\varepsilon$ that was applied in Theorem 7.1.13 and thus in Definition 7.2.7 is constant (for positive times $t \gtrsim \varepsilon$), which is the familiar concept one encounters in the study of singular SPDE as we have seen in Chapter 4. If δ^ε comes from wavelets such as the Haar basis, $\mathscr{C}^\varepsilon$ is usually not constant but a periodic function with a period (of order) ε (for positive times $t \gtrsim \varepsilon$). Thus we see that our analysis gives rise to a "non-constant renormalization". It is natural to ask if one can do with constant renormalization after all. For the sake of argument, consider $\mathscr{C}^\varepsilon$, periodic with period ε, with mean

$$C_\varepsilon = \frac{1}{\varepsilon}\int_K^{K+\varepsilon} \mathscr{C}^\varepsilon(t)dt.$$

with $K > 2c$, where $c > 0$ is as in Definition 7.1.5. From Lemma 7.1.12 it follows that $\mathscr{C}^\varepsilon$ (and its mean) are bounded by $\varepsilon^{H-1/2}$, uniformly in t. Putting all this together it easily follows that $|\langle \mathscr{C}^\varepsilon - C_\varepsilon \mathbf{1}_{[0,\infty)}, \varphi\rangle| \lesssim \varepsilon^{\alpha+H-1/2}$, uniformly over all φ bounded in $\mathcal{C}^\alpha$ for $\alpha \in (0,1)$, with convergence to zero when $\alpha > 1/2 - H$. As a consequence, taking $\varphi(t) = f(\hat{W}^\varepsilon)$, for smooth f, we clearly can apply this with any $\alpha < H$. Hence, by equating the constraints on α, we arrive at $H > 1/4$. The practical consequence then is, with focus on the convergence stated in part *(i)* of Theorem 7.40 that we can indeed replace non-constant renormalization by a constant, however at the prize of restricting to $H > 1/4$ and with an according loss on the convergence rate. Interestingly, the numerical simulation in [BFG$^+$17] suggest that no loss occurs and constant renormalization works for any $H > 0$. We have refrained from investigating this (technical) point further.

7.3 The case of the Haar basis

The following special case of the approximations above to $\int_0^t f(\hat{W}_r, r)\mathrm{d}W_r$ is of particular interest in [BFG$^+$17]. We here collect some more concrete formulas that arise in this case.

Let $\varepsilon = 2^{-N}$, $\phi := \mathbf{1}_{[0,1)}$ and $\phi_{l,N} = 2^{N/2}\phi(2^N \cdot - l), l \in \mathbb{Z}$ and the corresponding δ^ε coming from this wavelet is then for $x, y \in \mathbb{R}$.

$$\delta^\varepsilon(x,y) = \sum_{l\in\mathbb{Z}} \phi_{l,N}(x)\phi_{l,N}(y) = 2^N \mathbf{1}_{[\lfloor x2^N\rfloor 2^{-N}, (\lfloor x2^N\rfloor+1)2^{-N})}(y)$$

The mollified Volterra-kernel (7.21) then takes the form

$$
\begin{aligned}
\mathscr{K}^\varepsilon(u,v) &= \int_0^\infty \int_0^\infty \delta^\varepsilon(v,x_1)\delta^\varepsilon(x_1,x_2)K(u-x_2)\mathrm{d}x_1\mathrm{d}x_2 \\
&= \sqrt{2H}\cdot 2^N \int_{[\lfloor v2^N\rfloor 2^{-N},(\lfloor v2^N\rfloor+1)2^{-N}\wedge u)} |u-x|^{H-1/2}\mathbf{1}_{\lfloor v2^N\rfloor 2^{-N}\leqslant u}\,\mathrm{d}x \\
&= \frac{\sqrt{2H}}{1/2+H}2^N \\
&\times \left(|u-\lfloor v2^N\rfloor 2^{-N}|^{1/2+H} - |u-(\lfloor v2^N\rfloor+1)2^{-N}\wedge u)|^{1/2+H}\right)\mathbf{1}_{\lfloor v2^N\rfloor 2^{-N}\leqslant u}\,.
\end{aligned}
$$

A special role is played by *diagonal function* as a renormalization,

$$
\mathscr{C}^\varepsilon(t) = \mathscr{K}^\varepsilon(t,t) = \frac{\sqrt{2H}\,2^N}{1/2+H}|t-\lfloor t2^N\rfloor 2^{-N}|^{1/2+H}\,. \tag{7.42}
$$

We have moreover

$$
\begin{aligned}
\hat{W}^\varepsilon(t) &= \int_0^t K(t-r)\,\mathrm{d}W^\varepsilon_r = \sum_{l=0}^\infty Z_l\int_0^t K(t-r)\phi_{k,N}(r)\mathrm{d}r \\
&= \sum_{l=0}^\infty 2^{-N/2}\mathscr{K}^\varepsilon(t,l2^{-N})Z_l = \sum_{l=0}^{\lfloor t2^N\rfloor} 2^{-N/2}\mathscr{K}^\varepsilon(t,l2^{-N})Z_l\,,
\end{aligned}
$$

where $Z_l = \langle \dot{W},\phi_{l,N}\rangle$ are i.i.d. $N(0,1)$ variables. As approximation we can finally take $I^\varepsilon_{f,M}(t)$ from Definition 7.2.7 with partition $\{[t_l,t_{l+1})\} = \{[l2^{-N},(l+1)2^{-N}\wedge t)\}$ which gives us

$$
\begin{aligned}
I^\varepsilon_{f,M}(t) = &\sum_{l=0}^{\lceil t2^N\rceil-1}\sum_{m=0}^{M}\frac{1}{m!}\partial_1^m f(\hat{W}^\varepsilon_{t_l},t_l)2^{N/2}Z_l\int_{t_l}^{t_{l+1}}\left(\hat{W}^\varepsilon_r-\hat{W}^\varepsilon_{t_l}\right)^m\mathrm{d}r- \\
&-\sum_{m=1}^{M}\frac{1}{(m-1)!}\partial_1^m f(\hat{W}^\varepsilon_{t_l}),t_l)\int_{t_l}^{t_{l+1}}\mathscr{C}^\varepsilon(r)\left(\hat{W}^\varepsilon_r-\hat{W}^\varepsilon_{t_l}\right)^{m-1}\mathrm{d}r
\end{aligned}
$$

and

$$
\mathscr{I}^\varepsilon_f(t) = \sum_{l=0}^{\lceil t2^N\rceil-1}\int_{t_l}^{t_{l+1}}[2^{N/2}Z_l\cdot f(\hat{W}^\varepsilon_r,r)\,\mathrm{d}r - \mathscr{C}^\varepsilon(r)\,\partial_1 f(\hat{W}^\varepsilon_r,r)]\,\mathrm{d}r\,.
$$

As explained at the end of the last section, numerical simulations in [BFG+17] suggest that $\mathscr{C}^\varepsilon(r)$ in these formulas could be replaced by its local mean, the constant

$$
2^N\int_0^{2^{-N}}\mathscr{C}^\varepsilon(r)\,\mathrm{d}r = \frac{\sqrt{2H}}{(H+1/2)(H+3/2)}2^{N(1/2-H)}\,.
$$

7.4 Technical Results

Lemma 7.4.1. *We have $\mathbf{1}_{[0,1]} \in \mathcal{C}^1_1(\mathbb{R})$.*

Proof. Note that for $j > -1$ we can directly construct the primitive of $\Psi^j = \mathcal{F}^{-1}_{\mathbb{R}}\varphi_j$ as

$$\phi^j = 2^{-j} \cdot 2^j \phi_0(2^j \cdot)$$

where $\phi_0(x) = \int \mathrm{d}z\, e^{2\pi \imath x z} \frac{\varphi_0}{2\pi \imath z} \in \mathcal{S}(\mathbb{R})$. Thus writing

$$\Delta_j(\mathbf{1}_{[0,1]})(x) = \int \mathrm{d}u\, \Psi^j_{x-u} \mathbf{1}_{[0,1]}(u) = \phi^j(x-1) - \phi^j(x)\,,$$

we obtain indeed

$$\|\Delta_j(\mathbf{1}_{[0,1]})\|_{L^1(\mathbb{R}^d)} \lesssim 2^{-j}\,.$$

Since further $\int \mathrm{d}x\, |\Delta_{-1}\mathbf{1}_{[0,1]}(x)| \leqslant \int \mathrm{d}x \int \mathrm{d}u\, \mathbf{1}_{[0,1]}(u)\, |\Psi^{-1}_{x-u}| < \infty$, the claim is proved. □

Lemma 7.4.2. *For $a, b > 0$ and $\delta \in [0,1]$ we have for $x \notin [0,1)$*

$$|a^x - b^x| \leqslant 2^{1-\delta}|x|^\delta (a^{x-\delta} \vee b^{x-\delta}) \cdot |a-b|^\delta$$

and for $x \in (0,1)$

$$|a^x - b^x| \leqslant 2^{1-\delta}|x|^\delta (a^{(x-1)\delta} b^{x(1-\delta)} \vee b^{(x-1)\delta} a^{x(1-\delta)}) \cdot |a-b|^\delta\,.$$

Proof. This follows from interpolation between $|a^x - b^x| \leqslant |x| \sup_{z \in [a,b]} z^{x-1}|a-b| \leqslant |x| a^{x-1} \vee b^{x-1} |a-b|$ and $|a^x - b^x| \leqslant a^x + b^x \leqslant 2a^x \vee b^x$. □

Proof of Lemma 7.1.7 . Rewriting $\hat{W}^\varepsilon(t) = \sqrt{2H}\int_0^\infty \mathrm{d}W_u \int_0^\infty \mathrm{d}r\, \delta^\varepsilon(r,u)\, |t-r|^{H-1/2}\mathbf{1}_{r<t}$ we have

$$\begin{aligned}
\mathbb{E}\left|\hat{W}^\varepsilon_t - \hat{W}^\varepsilon_s\right|^2 &= 2H \int_0^\infty \mathrm{d}u \left(\int_0^\infty \mathrm{d}r \delta^\varepsilon(r,u)(\mathbf{1}_{r<t}|t-r|^{H-1/2} - \mathbf{1}_{r<s}|s-r|^{H-1/2}) \right)^2 \\
&\lesssim \int_0^\infty \mathrm{d}u \int_0^\infty \mathrm{d}r |\delta^\varepsilon(r,u)| \left(\mathbf{1}_{r<t}|t-r|^{H-1/2} - \mathbf{1}_{r<s}|s-r|^{H-1/2}\right)^2 \\
&\lesssim \int_0^{s\vee t} \mathrm{d}r \left(\mathbf{1}_{r<t}|t-r|^{H-1/2} - \mathbf{1}_{r<s}|s-r|^{H-1/2}\right)^2\,,
\end{aligned}$$

where we used the Itô isometry in the first and Jensen's inequality in the second step. Assuming $s < t$ we can split the integral in domains $[0,s]$ and $[s,t]$ which yields the bound $|t-s|^{2H}\int_0^s |s-r|^{4H-1} + |t-s|^{2H} \lesssim |t-s|^{2H}$. Application of equivalence of moments for Gaussian random variables and Kolmogorov's criterion then shows the first inequality.

The second one follows by interpolation (and once more Kolmogorov) if we can prove that

$$\mathbb{E}|\hat{W}_t^\varepsilon - \hat{W}_t|^2 \lesssim \varepsilon^{2H-\kappa'}. \tag{7.43}$$

We have, by Itô's isometry,

$$\mathbb{E}\left|\hat{W}_t^\varepsilon - \hat{W}_t\right|^2 = 2H\int_0^\infty \mathrm{d}u \left(\int_0^\infty \mathrm{d}r\, \delta^\varepsilon(r,u)\, |t-r|^{H-1/2}\mathbf{1}_{r<t} - |t-u|^{H-1/2}\mathbf{1}_{u<t}\right)^2 .$$

We can enlarge the inner integral such that $\int \delta^\varepsilon(r,u) = 1$ by negleting an error term which can be estimated by $\int_{B(0,c\varepsilon)} \mathrm{d}u (\int_{B(0,c\varepsilon)} \mathrm{d}r\, \varepsilon^{-1}|t-r|^{H-1/2})^2 \lesssim \varepsilon^{2H}$. Application of Jensen's inequality then yields

$$\int_0^\infty \mathrm{d}u \int_{-\infty}^\infty \mathrm{d}r\, |\delta^\varepsilon(r,u)| \left(|t-r|^{H-1/2}\mathbf{1}_{r<t} - |t-u|^{H-1/2}\mathbf{1}_{u<t}\right)^2 .$$

The cases where either $r > u$ or $u > t$ yield an ε^{2H} error term as above so that bounding with Lemma 7.4.2

$$\left||t-r|^{H-1/2} - |t-u|^{H-1/2}\right| \lesssim (|t-r|^{-1/2+\kappa} + |t-u|^{-1/2+\kappa}) \cdot |u-r|^{H-\kappa}$$

proves (7.43). □

Lemma 7.4.3. *For c as in Definition 7.1.5 and $t > 2c\varepsilon$ and $s \in \mathbb{R}$ we have for $\kappa' \in (0,H)$*

$$|K(s-t) - \mathscr{K}^\varepsilon(s,t)| \lesssim |s-t|^{H-1/2-\kappa'}\varepsilon^{\kappa'} .$$

Proof. If $2c\varepsilon \geqslant |s-t|/2$ the bound easily follows from Lemma 7.1.12. If $2c\varepsilon \geqslant |s-t|/2$ we can reshape

$$|K(s-t) - \mathscr{K}^\varepsilon(s,t)| = \left|\int_{-\infty}^\infty \mathrm{d}u\, \delta^{2,\varepsilon}(t,u)(\mathbf{1}_{t<s}|s-t|^{H-1/2} - \mathbf{1}_{s<u}|s-u|^{H-1/2})\right| ,$$

where $\delta^{2,\varepsilon}(t,\cdot) := \int_{-\infty}^\infty \mathrm{d}x_1 \int_{-\infty}^\infty \mathrm{d}x_2 \delta^\varepsilon(t,x_1)\delta^\varepsilon(x_1,\cdot)$ satisfies the properties in Definition 7.1.5 with support in $B(t,2c\varepsilon)$. Note that for $2c\varepsilon \geqslant |s-t|/2$ either both indicator functions vanish or none so that we only have to consider $t < s$ where we obtain with Lemma 7.4.2 up to a constant $\int_{-\infty}^\infty |\delta^{2,\varepsilon}(t,u)||t-s|^{H-1/2-\kappa'}\varepsilon^{\kappa'} \lesssim |t-s|^{H-1/2-\kappa'}\varepsilon^{\kappa'}$. □

Lemma 7.4.4. *For $F \in L^2(\mathbb{P} \times \mathrm{Leb})$ we have*

$$\mathbb{E}\left[\left|\int F(t)\mathrm{d}W_t^\varepsilon\right|^2\right] \lesssim \int \mathbb{E}\left[|F(t)|^2\right]\mathrm{d}t$$

Proof. As a consequence of Definition 7.1.5, we have $\int |\delta^\varepsilon(x,y)\mathrm{d}x|$ is bounded uniformly in ε and y. We can, therefore, normalize $|\delta^\varepsilon(\cdot,r)|$ to a probability density and apply Itô's isometry and Jensen's inequality to

$$\int F(t)\mathrm{d}W_t^\varepsilon = \int_0^\infty \int_0^\infty \delta^\varepsilon(t,r)\, F(t)\mathrm{d}t\, \mathrm{d}W_r.$$

$\square$

Chapter 8

The nonlinear Schrödinger equation on the full space

Let us recall that the stochastic nonlinear Schrödinger equation (SNLS) in dimension 2 is given by

$$\imath \partial_t u = \Delta u + \lambda u |u|^{2\sigma} + u\,\xi,\ u(0) = u_0 \tag{8.1}$$

for $u : [0, T] \times \mathbb{R}^2 \to \mathbb{C}$ with white noise in space $\xi \in \mathcal{S}'(\mathbb{R}^2)$, as in Definition 2.2.1. Relation (8.1) can be seen as the stochastic version of the well-studied deterministic nonlinear Schrödinger equation (compare for example [Caz03], [Kat87], [Caz79], [BG80]), where the noise term $u\,\xi$ is absent.

While the deterministic equation arises in nonlinear optics to model laser propagation in a dispersive material [Ber98, Section 1.1.-1.3.], the stochastic term can be seen as taking into account disorder in the considered medium. The deterministic parameters $\sigma > 0$ and $\lambda \in \mathbb{R}$ represent in this context material constants. The case $\sigma < 1$ is known as the *subcritical* regime and $\sigma \geqslant 1$ as *(super-)critical.* For a positive coefficient $\lambda > 0$ the equation is called *focusing*, while for $\lambda < 0$ one uses the term *defocusing*.

The existence of solutions for (8.1) on a periodic setup $[0, T] \times \mathbb{T}^2$ ($\mathbb{T}^2 = [-\frac{1}{2}, \frac{1}{2})^2$ denoting the torus in dimension 2) was recently studied in [DW16], where the authors prove global existence of solutions for $\sigma = 1$ (and suitable λ). Their proof can be easily modified to show that (8.1) has global solutions in a periodic set-up for $\sigma \in (0, 1)$ (and $\lambda \in \mathbb{R}$).

In this chapter we show that the equation (8.1) does in fact possess solutions on the full space $\mathbb{R}^2$ in the subcritical regime $\sigma \in (0, 1)$ (or if (8.1) is defocusing). More precisely we show the following meta-theorem.

Theorem 8.0.1. *Under suitable initial conditions* u_0 (8.1) *has a unique local solution on* $[0, T] \times \mathbb{R}^2$ *for some random time* $T > 0$ *if either* $\sigma < 1$ *or* $\lambda \leqslant 0$ *is satisfied. If* $\sigma < 1/2$ *(or* $\lambda = 0$*) the solution is global.*

This statement is shown below in the two main results of this chapter, Theorem 8.4.4 and 8.5.2. As in [DW16] we rely on a transformation of the equation that stems originally from [HL15] and make use of conservation of mass and energy to prove a priori bounds for (8.1). An application of these methods is more delicate on the full space since one needs control over the decay of the solution to counterbalance the growth of the noise. A key role in this task will be played by Lemma 8.3.1 below which allows us to trade some differentiability of the solution against some localization on compact time intervals. Most of this chapter (including parts from this introduction) is taken from [DM17].

The technical background for this section was layed in Chapter 2, where weighted Besov spaces were discussed. Similar as in Chapter 4 we only measure the spatial smoothness of the solutions in terms of these spaces, so that in particular we choose isotropic scaling

$$\mathfrak{s} = (1, 1)\,.$$

In Section 8.1 we recall and discuss a few properties of weighted Besov spaces, which we also presented in Chapter 2. We further introduce the fundamental quantities which we need for the proof of Theorem 8.0.1 above. Roughly speaking, Section 8.3 is then devoted to the control of the growth of u (or rather its transformed analogue), while Sections 8.4 and 8.5 show an H^2 bound which allows for a solution of (8.1) in Theorems 8.4.4 and 8.5.2.

Constants

Random constants will be denoted in this chapter by K. $(K_\varepsilon)_{\varepsilon\in(0,1]}$ denotes a family of random constants with $L^p(\mathbb{P})$-norms bounded independent of ε for all $p \in [1, \infty)$. We also write $K_k := K_{2^{-k-1}} \cdot K_{2^{-k}}$ and indicate further by this notation that also this sequence is bounded *almost surely* in $k = 0, 1, 2, \ldots$. Whenever we use random constants such as K, K_ε, K_k or deterministic constants (which will only have the symbols C and a in this chapter) we always allow them to change from line to line.

8.1 Techniques

8.1.1 Estimates on weighted Besov spaces

We only work with polynomial weights $\rho \in \boldsymbol{\rho}(\omega^{\mathrm{pol}})$ for the considered Besov spaces $\mathcal{B}^\gamma_{p,q}(\mathbb{R}^d, \rho)$ in this chapter, more precisely we will always take weights $\rho(x)$ in the form

$$\rho(x) = \langle x\rangle^\mu = (1 + |x|^2)^{\frac{\mu}{2}}\,,$$

where $\mu \in \mathbb{R}$ can be positive, negative or 0. The motivation to take the smoothened weights instead of, say, $(1 + |x|)^\mu$ is that a multiplication by $\langle x\rangle^\mu$ does not change the smoothness of a function (or distribution) f. In fact it can be showed [Tri06, Theorem 6.5] that the weights $\langle x\rangle^\mu$ can be "pulled inside the Besov norm"

$$\|f\|_{\mathcal{B}^\gamma_{p,q}(\mathbb{R}^d,\langle x\rangle^\mu)} \approx \|f\langle x\rangle^\mu\|_{\mathcal{B}^\gamma_{p,q}(\mathbb{R}^d)} . \tag{8.2}$$

We will work in this chapter solely with (isotropic) Besov spaces $\mathcal{B}^\gamma_{p,q}(\mathbb{R}^d, \rho)$ with $p = q$ for which we write shorthand

$$\mathcal{B}^\gamma_p(\mathbb{R}^d, \rho) := \mathcal{B}^\gamma_{p,p}(\mathbb{R}^d, \rho) ,$$

with the usual convention that we drop ρ in the parantheses whenever $\rho = 1$. Let us recall the following identities from Chapter 2:

$$\begin{aligned} H^\gamma(\mathbb{R}^d, \langle x\rangle^\mu) &= \mathcal{B}^\gamma_2(\mathbb{R}^d, \langle x\rangle^\mu) , \\ \mathcal{C}^\gamma(\mathbb{R}^d, \langle x\rangle^\mu) &= \mathcal{B}^\gamma_\infty(\mathbb{R}^d, \langle x\rangle^\mu) . \end{aligned}$$

where $H^\gamma(\mathbb{R}^d, \langle x\rangle^\mu)$ can be identified with the *weighted* Bessel potential space given by those f for which $\|\mathcal{F}^{-1}_{\mathbb{R}^d}\langle\cdot\rangle^\gamma\mathcal{F}_{\mathbb{R}^d} f\|_{L^2(\mathbb{R}^d,\langle x\rangle^\mu)} < \infty$, compare for example [Sch09, Section 3.1]. For $\gamma \in \mathbb{N}$ the space $H^\gamma(\mathbb{R}^d, \langle x\rangle^\mu)$ just coincides with the (maybe more familiar) weighted Sobolev space with equivalent norm [Tri06, Thm. 6.9]

$$\|f\|_{H^\gamma(\mathbb{R}^d,\langle x\rangle^\mu)} \approx \sum_{k\in\mathbb{N}^d :\, |k|\leqslant\gamma} \|\partial^k f\|_{L^2(\mathbb{R}^d,\langle x\rangle^\mu)} . \tag{8.3}$$

Note that in particular $\mathcal{B}^0_2(\mathbb{R}^d, \langle x\rangle^\mu) = H^0(\mathbb{R}^d, \langle x\rangle^\mu) = L^2(\mathbb{R}^d, \langle x\rangle^\mu)$.

$\mathcal{C}^\gamma(\mathbb{R}^d, \langle x\rangle^\mu)$, we recall, is the weighted Hölder-Zygmund spaces that coincides with Hölder spaces for $\gamma \in \mathbb{R}_+\backslash\mathbb{N}$ (Lemma 2.1.23). One easily sees that the norm in Lemma 2.1.23 can be bounded for $\mu \in \mathbb{R}$, $\gamma \in \mathbb{R}_+\backslash\mathbb{N}$ by

$$\|f\|_{\mathcal{C}^\gamma(\mathbb{R}^d,\langle x\rangle^\mu)} \lesssim \sup_{k\in\mathbb{N}} \langle k\rangle^\mu \, \|f\|_{\mathcal{C}^\gamma([-k,k]^2)} \tag{8.4}$$

for $\mu \in \mathbb{R}$ (with equivalence if $\mu \leqslant 0$), where $\|f\|_{\mathcal{C}^\gamma([-k,k]^2)}$ is defined as in Remark 2.1.25.

We mostly work with the spaces $H^\gamma(\mathbb{R}^d, \langle x\rangle^\mu)$, $\mathcal{C}^\gamma(\mathbb{R}^d, \langle x\rangle^\mu)$ and use the class of weighted Besov spaces $\mathcal{B}^\gamma_p(\mathbb{R}^d, \langle x\rangle^\mu)$ as a framework that connects these sets. Let us summarize a few properties of this class which we have (mostly) already seen in Chapter 2.

Lemma 8.1.1. *We have the following properties of weighted Besov spaces.*

(i) *(Besov embedding) For* γ_1, γ_2, μ_1, $\mu_2 \in \mathbb{R}$ *and* p_1, $p_2 \in [1,\infty]$ *with* $\mu_1 \leqslant \mu_2$, $\gamma_1 - \frac{d}{p_1} \leqslant \gamma_2 - \frac{d}{p_2}$, $\gamma_1 \leqslant \gamma_2$, *we have the continuous embedding*

$$\mathcal{B}^{\gamma_2}_{p_2}(\mathbb{R}^d, \langle x\rangle^{\mu_2}) \subseteq \mathcal{B}^{\gamma_1}_{p_1}(\mathbb{R}^d, \langle x\rangle^{\mu_1}) \,.$$

(ii) *(Sobolev embedding) For* $\gamma > 0$, $p \in [2,\infty)$ *such that* $-\frac{d}{p} \leqslant \gamma - \frac{d}{2}$ *and* $\mu_1, \mu_2 \in \mathbb{R}$, $\mu_1 \leqslant \mu_2$ *we have the continuous embedding*

$$H^{\gamma}(\mathbb{R}^d, \langle x\rangle^{\mu_2}) \subseteq L^p(\mathbb{R}^d, \langle x\rangle^{\mu_1}) \,.$$

(iii) *(Duality) For* $\gamma \in \mathbb{R}$, $p \in [1,\infty)$, $\mu \in \mathbb{R}$ *and* $1/p' := 1 - 1/p$ *we have the duality*

$$\big(\mathcal{B}^{\gamma}_{p}(\mathbb{R}^d, \langle x\rangle^{\mu})\big)' = \mathcal{B}^{-\gamma}_{p'}(\mathbb{R}^d, \langle x\rangle^{-\mu}) \,.$$

(iv) *(Multiplication) For* μ_1, $\mu_2 \in \mathbb{R}$, p_1, $p_2 \in [1,\infty]$ *and* γ_1, $\gamma_2 \in \mathbb{R}$ *with* $\gamma_1 + \gamma_2 > 0$ *we have for* $\gamma = \gamma_1 \wedge \gamma_2 \wedge (\gamma_1 + \gamma_2)$, $1/p := 1/p_1 + 1/p_2$, $\mu = \mu_1 + \mu_2$ *and any* $\kappa > 0$

$$\Big\| f_1 \cdot f_2 \|_{\mathcal{B}^{\gamma-\kappa}_{p}(\mathbb{R}^d, \langle x\rangle^{\mu})} \lesssim \| f_1 \|_{\mathcal{B}^{\gamma_1}_{p_1}(\mathbb{R}^d, \langle x\rangle^{\mu_1})} \| f_2 \|_{\mathcal{B}^{\gamma_2}_{p_2}(\mathbb{R}^d, \langle x\rangle^{\mu_2})} \,.$$

(v) *(Interpolation) For* $p_0, p_1 \in [1,\infty]$, μ_0, μ_1, γ_0, $\gamma_1 \in \mathbb{R}$ *and* p, μ, γ *such that* $1/p = (1-\Theta)/p_0 + \Theta/p_1$, $\mu = (1-\Theta)\mu_0 + \Theta\mu_1$ *and* $\gamma = (1-\Theta)\gamma_0 + \Theta\gamma_1$ *for some* $\Theta \in [0,1]$ *we have*

$$\| f \|_{\mathcal{B}^{\gamma}_{p}(\mathbb{R}^d, \langle x\rangle^{\mu})} \leqslant \| f \|^{1-\Theta}_{\mathcal{B}^{\gamma_0}_{p_0}(\mathbb{R}^d, \langle x\rangle^{\mu_0})} \| f \|^{1-\Theta}_{\mathcal{B}^{\gamma_1}_{p_1}(\mathbb{R}^d, \langle x\rangle^{\mu_1})} \,.$$

Proof. Property *(i)* is a simplification of Lemma 2.1.26, *(iii)* of Lemma 2.1.29, *(iv)* of Corollary 2.1.35 and *(v)* is a simplification of Lemma 2.1.31. A quick way to see *(ii)* is to apply the analogue of *(i)* for weighted Triebel-Lizorkin spaces [Tri06, Theorem 6.7, Theorem 6.9]. □

An economic way to remember property *(i)* and *(ii)* is to introduce the *Sobolev number*

$$\mathrm{Sob}(\mathcal{B}^{\gamma}_{p}(\mathbb{R}^d, \rho)) := \gamma - \frac{d}{p} \tag{8.5}$$

for $\gamma \in \mathbb{R}$, $p \in [1,\infty]$ and $\rho \in \boldsymbol{\rho}(\boldsymbol{\omega})$. The number $\mathrm{Sob}(\mathcal{B}^{\gamma}_{p}(\mathbb{R}^d, \rho)) \in \mathbb{R}$ quantifies some hybrid of integrability and smoothness of the space $\mathcal{B}^{\gamma}_{p}(\mathbb{R}^d, \rho)$. We also introduce the Sobolev number for L^p-spaces in the same spirit

$$\mathrm{Sob}(L^p(\mathbb{R}^d, \rho)) := 0 - \frac{d}{p} = -\frac{d}{p} \,. \tag{8.6}$$

for $p \in [1,\infty]$ and $\rho \in \boldsymbol{\rho}(\boldsymbol{\omega})$. The conditions for the continuous embeddings *(i)* and *(ii)* can then be reformulated as

(i) $\mathrm{Sob}(\mathcal{B}^{\gamma_1}_{p_1}(\mathbb{R}^d, \langle x \rangle^{\mu_1})) \leqslant \mathrm{Sob}(\mathcal{B}^{\gamma_2}_{p_2}(\mathbb{R}^d, \langle x \rangle^{\mu_2}))$ and $\gamma_1 \leqslant \gamma_2$, $\mu_1 \leqslant \mu_2$.

(ii) $\mathrm{Sob}(L^p(\mathbb{R}^d, \langle x \rangle^{\mu_1}) \leqslant \mathrm{Sob}(H^\gamma(\mathbb{R}^d, \langle x \rangle^{\mu_2}))$ and $\gamma > 0, 2 \leqslant p < \infty$, $\mu_1 \leqslant \mu_2$.

Note that the Sobolev number of $L^p(\mathbb{R}^d, \langle x \rangle^{\mu_1})$, that is $-d/p$, is always negative as long as $p < \infty$. In particular, a space $H^\gamma(\mathbb{R}^d, \langle x \rangle^{\mu_2})$ with Sobolev number 0 or equivalently

$$\gamma = \frac{d}{2}$$

contains any $L^p(\mathbb{R}^d, \langle x \rangle^{\mu_2})$ with $\mu_1 \leqslant \mu_2$, $p \in [2, \infty)$. The *Brezis-Gallouet inequality* states that the critical space $H^{\frac{d}{2}}(\mathbb{R}^d, \langle x \rangle^{\mu_1})$ further "almost contains" $L^\infty(\mathbb{R}^d, \langle x \rangle^{\mu_2})$, $\mu_1 \leqslant \mu_2$.

Lemma 8.1.2 (Brezis-Gallouet inequality)**.** *For $\mu \in \mathbb{R}$ and $\gamma > 0$ it holds*

$$\|f\|_{L^\infty(\mathbb{R}^d, \langle x \rangle^\mu)} \lesssim (1 + \|f\|_{H^{\frac{d}{2}}(\mathbb{R}^d, \langle x \rangle^\mu)}) \sqrt{1 + \log(1 + \|f\|_{\mathcal{C}^\gamma(\mathbb{R}^d, \langle x \rangle^\mu)})}$$

Remark 8.1.3. *The Brezis-Gallouet inequality was first stated in [BG80] (also in the context of a nonlinear Schrödinger equation). We here cited a slightly more general version from [Oza95, Thm. 2], or rather the one actually shown in the proof of Theorem 2 in [Oza95] (skipping the step of the Sobolev embedding $H^{d/q+,q} \subseteq \mathcal{C}^\gamma$ used in the reference). The version proved there is withough weigths, but we get then easily the statement above by applying* (8.2).

Finally let us mention the elementary *Young product inequality*, we use occasionally to close estimates: For x, y, $\varepsilon > 0$, $\delta \in (0, 1)$ it holds

$$x^{1-\delta} \cdot y^\delta \leqslant \varepsilon\, x + C_\delta\, \varepsilon^{-\frac{1-\delta}{\delta}} y \tag{8.7}$$

with $C_\delta = (1 - \delta)\, \delta^{\frac{1-\delta}{\delta}}$ (this is a scaled version of [Yos74, Lemma I.3.1]).

8.1.2 Growth of the stochastic data

As already pointed out in the introduction of this thesis the main term of (8.1) that causes difficulties is the ill-defined product

$$u \cdot \xi\,.$$

The key insight in [DW16] was that a transformation of u yields a better behaved equation that can be controlled via conserved quantities. This transformation was first applied in [HL15] to solve the parabolic Anderson model 1.8. As we have seen in Chapter 4 the solution to the latter can be smoothened by subtracting a term (a paraproduct) that behaves on small scales like the solution Y to the Poisson equation with forcing ξ. It might therefore be not surprising that the transformation in [HL15] involves exactly this quantity, more precisely one defines

$$v = e^Y u$$

which transform (8.1) into a "better" equation for v as we will see below. Our definition of Y is here slightly distinct from Chapter 4. We essentially proceed as in [HL15] and use a truncated Green's function $G \in C^\infty(\mathbb{R}^2 \backslash \{0\})$ that satisfies $\operatorname{supp} G \subseteq B(0,1)$ and $G(x) = \frac{1}{2\pi} \log |x|$ for $|x|$ small enough, so that $Y := G * \xi$ solves

$$\Delta Y = \xi + \varphi * \xi$$

for some $\varphi \in C_c^\infty(\mathbb{R}^2)$. A special role in this chapter will be played by the Wick product

$$\nabla Y^{\diamond 2} .$$

The gradient ∇Y is a distribution, so that we have to say what we mean by $\nabla Y^{\diamond 2}$. We already pointed out in Remark 7.1.1 that the Skorohod integral can be read as some sort of distributional Wick product, so that it is quite natural to define

$$\nabla Y^{\diamond 2}(\varphi) := \int \xi(\mathrm{d}u) \diamond \int \xi(\mathrm{d}v) \int \mathrm{d}x \, \varphi(x) \, \nabla G(x-u) \cdot \nabla G(x-v) \, ,$$

where $\varphi \in \mathcal{S}(\mathbb{R}^2)$ and where $\diamond$ indicates the Skorohod integral w.r.t to the Gaussian measure $\xi(\mathrm{d}u)$ on $\mathbb{R}^2$ induced by ξ (as in [Jan97]). We will use the following Lemma from [HL15].

Lemma 8.1.4. *For $\delta > 0$, $\alpha \in (0,1)$ and $p \in [1, \infty)$ we have*

$$\mathbb{E}[\|Y\|^p_{\mathcal{C}^\alpha(\mathbb{R}^2, \langle x \rangle^{-\delta})} + \|\nabla Y^{\diamond 2}\|^p_{\mathcal{C}^{\alpha-1}(\mathbb{R}^2, \langle x \rangle^{-\delta})}] < \infty \, .$$

As in Section 2.2 the polynomial weight is in fact more than we need since the noise, and thus Y, $\nabla Y^{\diamond 2}$, grow actually like $\sqrt{\log(x)}$ (recall that we used a compactly supported Green's function G). This fact allows us to prove the following bound on e^Y.

Corollary 8.1.5. *For any $a \in \mathbb{R}$, $\alpha \in (0,1)$, $p \in [1, \infty)$ and $\delta > 0$ we have*

$$\mathbb{E}[\|e^{aY}\|^p_{\mathcal{C}^\alpha(\mathbb{R}^2, \langle x \rangle^{-\delta})}] < \infty \, .$$

Proof. Note first that we can bound

$$\|e^{aY}\|_{\mathcal{C}^\alpha(\mathbb{R}^2,\langle x\rangle^{-\delta})} \overset{(8.4)}{\lesssim} \sup_{k\in\mathbb{N}} \frac{\|e^{aY}\|_{\mathcal{C}^\alpha([-k,k]^2)}}{k^\delta} \leqslant \sup_{k\in\mathbb{N}} \frac{\exp(C|a|\|Y\|_{\mathcal{C}^\alpha([-k,k]^2)})}{k^\delta},$$

for some deterministic constant $C > 0$. One can see the last inequality for example by expanding the exponential in its series and using that $\mathcal{C}^\alpha([-k,k]^2)$ is an algebra for $\alpha \in (0,1)$ (with involved constants independent of k). It remains to bound the p-th moment of the right hand side. Using the compact support of the Green's function we see

$$\|Y\|_{\mathcal{C}^\alpha([-k,k]^2)} \lesssim \|\chi_{k+2}\xi\|_{\mathcal{C}^{\alpha-2}(\mathbb{R}^2))}$$

(to show this one can for example use the wavelet characterization of Besov spaces, as in [Tri06], and a decomposition of G as in [Hai14, Remark 5.6]). With Lemma 2.2.4 we can therefore find λ, $\lambda' > 0$ such that

$$\sup_{k\in\mathbb{N}} \frac{\mathbb{E}\left[\exp(\lambda\|Y\|^2_{\mathcal{C}^\alpha([-k,k]^2)})\right]}{k^{\lambda'}} < \infty .$$

We now pick, without loss of generality, $p \in [1,\infty)$ so big that $p\cdot\delta \geqslant 2+\lambda'$ which gives us

$$\begin{aligned}\mathbb{E}\left[\left|\sup_{k\in\mathbb{N}} \frac{\exp(C|a|\|Y\|_{\mathcal{C}^\alpha([-k,k]^2)})}{k^\delta}\right|^p\right] &\leqslant \sum_{k=1}^\infty \frac{\mathbb{E}\left[\exp(pC|a|\|Y\|_{\mathcal{C}^\alpha([-k,k]^2)}\right]}{k^{\delta p}} \\ &\lesssim \sum_{k=1}^\infty \frac{\mathbb{E}\left[\exp(\lambda\|Y\|^2_{\mathcal{C}^\alpha([-k,k]^2)}\right]}{k^2\cdot k^{\lambda'}} < \infty ,\end{aligned}$$

where we used $\sup_{x\geqslant 0} e^{pC|a|x-\lambda x^2} < \infty$ in the last step. The proof is finished. □

Using once more (8.4), the compact support of φ and Lemma 2.2.2 one also derives the bound

$$\mathbb{E}\left[\|\varphi * \xi\|^p_{\mathcal{C}^\beta(\mathbb{R}^2,\langle x\rangle^{-\delta})}\right] < \infty$$

for any $\beta \in \mathbb{R}$, $\delta > 0$ and $p \in [1,\infty)$. We will mostly work with smoothened noise, so fix from now on to the end of this chapter a mollifier $\rho \in C_c^\infty(B(0,1))$ and define for $\rho_\varepsilon := \varepsilon^{-2}\rho(\varepsilon^{-1}\cdot)$, $\varepsilon \in (0,1]$

$$\xi_\varepsilon = \rho_\varepsilon * \xi,\ Y_\varepsilon = G * \xi_\varepsilon .$$

and we have once more (with the same $\varphi \in C_c^\infty(\mathbb{R}^2)$ as above)

$$\Delta Y_\varepsilon = \xi_\varepsilon + \varphi * \xi_\varepsilon \,.$$

Using that $\operatorname{supp} \rho_\varepsilon \subseteq B(0,1)$ all the growth results above can be carried over and we obtain that for any $\delta > 0, \alpha \in (0,1)$, $\beta \in \mathbb{R}$ and $a \in \mathbb{R}$ we have the following bound

$$\|Y_\varepsilon\|_{\mathcal{C}^\alpha(\mathbb{R}^2,\langle x\rangle^{-\delta})} + \|\nabla Y_\varepsilon^{\diamond 2}\|_{\mathcal{C}^{\alpha-1}(\mathbb{R}^2,\langle x\rangle^{-\delta})} + \|e^{aY_\varepsilon}\|_{\mathcal{C}^\alpha(\mathbb{R}^2,\langle x\rangle^{-\delta})} + \|\varphi * \xi_\varepsilon\|_{\mathcal{C}^\beta(\mathbb{R}^2,\langle x\rangle^{-\delta})} \leqslant K_\varepsilon \,, \tag{8.8}$$

where we recall that K_ε denotes in this chapter a (changing) random constant that is bounded in $L^p(\mathbb{P})$ for $p \in [1,\infty)$, independent of $\varepsilon \in (0,1]$. Note that we can read the Wick product $\nabla Y_\varepsilon^{\diamond 2}$ once more as a Skorohod integral or, since ∇Y_ε is a genuine function, as a classical Wick product of Gaussian random variables

$$\nabla Y_\varepsilon^{\diamond 2} = \nabla Y_\varepsilon^2 - \mathbb{E}[|\nabla Y_\varepsilon|^2] \,. \tag{8.9}$$

Using the definition of white noise one sees that the difference

$$\nabla Y_\varepsilon^2 - \nabla Y_\varepsilon^{\diamond 2} = \mathbb{E}[|\nabla Y_\varepsilon|^2] = \int \mathrm{d}x \, |(\nabla \rho_\varepsilon * G)(x)|^2 \approx |\log \varepsilon|^2 \tag{8.10}$$

diverges, indicating that the "usual product" $\nabla Y^2 = \nabla Y \cdot \nabla Y$ is ill-defined. We will also use the following statements, which we again take from [HL15]:

Lemma 8.1.6. *For $\alpha \in (0,1)$ and $\kappa \in (0, 1-\alpha)$ we have*

$$\|Y_\varepsilon - Y\|_{\mathcal{C}^\alpha(\mathbb{R}^2,\langle x\rangle^{-\delta})} + \|\nabla Y_\varepsilon^{\diamond 2} - \nabla Y^{\diamond 2}\|_{\mathcal{C}^{\alpha-1}(\mathbb{R}^2,\langle x\rangle^{-\delta})} \leqslant K_\varepsilon \varepsilon^\kappa \,. \tag{8.11}$$

Together with the bounds (8.8) and Corollary 8.1.5 we then obtain

$$\|e^{aY} - e^{aY_\varepsilon}\|_{\mathcal{C}^\alpha(\mathbb{R}^2,\langle x\rangle^{-\delta})} \leqslant K_\varepsilon \, \varepsilon^\kappa \tag{8.12}$$

for $a \in \mathbb{R}$, $\alpha \in (0,1)$ and $\kappa \in (0, 1-\alpha)$. Further we have for $\beta \in \mathbb{R}$, $\alpha \in (0,1)$ $\delta > 0$ and $\kappa \in (0, 1-\alpha)$

$$\|\varphi * \xi_\varepsilon - \varphi * \xi\|_{\mathcal{C}^\beta(\mathbb{R}^2,\langle x\rangle^{-\delta})} \lesssim \|\xi - \xi_\varepsilon\|_{\mathcal{C}^\alpha(\mathbb{R}^2,\langle x\rangle^{-\delta})} \leqslant K_\varepsilon \varepsilon^\kappa \,, \tag{8.13}$$

where we used in the last step $\|\xi - \xi_\varepsilon\|_{\mathcal{C}^\alpha(\mathbb{R}^2,\langle x\rangle^{-\delta})} \lesssim K_\varepsilon \varepsilon^\kappa$ due to [HL15, Lemma 1.1].

It will turn out convenient to have an estimate on the blow-up of the L^p norm of ∇Y, which is covered by the following Lemma.

Lemma 8.1.7. *Given $\delta \in (0,1)$, $p \in (2/\delta, \infty)$ and $q \in [p, \infty)$ we have*

$$\mathbb{E}\left[\|\nabla Y_\varepsilon\|^q_{L^p(\mathbb{R}^2,\langle x\rangle^{-\delta})}\right] \lesssim |\log(\varepsilon)|^{q/2} \text{ and } \mathbb{E}\left[\|\nabla Y_\varepsilon^{\diamond 2}\|^q_{L^p(\mathbb{R}^2,\langle x\rangle^{-\delta})}\right] \lesssim |\log(\varepsilon)|^q \,.$$

Proof. We estimate via Jensen's inequality

$$\mathbb{E}\left[\|\nabla Y_\varepsilon\|^q_{L^p(\mathbb{R}^2,\langle x\rangle^{-\delta})}\right] \lesssim \int_{\mathbb{R}^2} \frac{1}{\langle x\rangle^{p\delta}}\,\mathbb{E}[|\nabla Y(x)|^q]\,\mathrm{d}x\,.$$

The first inequality follows now from equivalence of moments for Gaussian random variables and the estimate $\mathbb{E}[|\nabla Y_\varepsilon(x)|^2] \lesssim |\log\varepsilon|$ (compare [HL15]). The second inequality can be proved via the same argument □

8.2 Setup and conserved quantities

We want to consider the equation

$$\imath\partial_t u = \Delta u + u\xi + \lambda|u|^{2\sigma}u,\ u(0) = u_0\,, \tag{8.14}$$

with a suitable renormalization we introduce below. It is well-known (see for example [Caz03]) that a solution to this equation, if existent, has at least formally the conserved quantities mass $N(u(t)) = N(u_0)$ and energy $H(u(t)) = H(u_0)$, defined as

$$N(u) = \int_{\mathbb{R}^2} |u|^2\,\mathrm{d}x,\ H(u) = \int_{\mathbb{R}^2} \frac{1}{2}|\nabla u|^2 - \frac{1}{2}|u|^2\xi - \frac{\lambda}{2\sigma+2}|u|^{2\sigma+2}\,\mathrm{d}x\,,$$

where we evaluate $u = u(t)$ at some time $t \in [0,T]$ which we suppressed in the notation to shorten the formulas a bit. We further did not write explicitly the dependency on the integration variable x.

As already pointed out above, we follow [DW16] in the idea to substitute u in this equation by $v = e^Y u$ and obtain the equivalent problem

$$\imath\partial_t v = \Delta v + v\,(\nabla Y^2 - \varphi * \xi) - 2\nabla v \cdot \nabla Y + |v|^{2\sigma}ve^{-2\sigma Y},\ v(0) = v_0 := e^{-Y}u_0\,.$$

As explained in [DW16] there is only hope to obtain a solution to this equation if we replace the square ∇Y^2 by a different expression for which we take the Wick product $\nabla Y^{\diamond 2}$.

$$\imath\partial_t v = \Delta v + v\,(\nabla Y^{\diamond 2} - \varphi * \xi) - 2\nabla v \cdot \nabla Y + \lambda|v|^{2\sigma}ve^{-2\sigma Y},\ v(0) = v_0\,. \tag{8.15}$$

We have seen in (8.9) that, on the level of approximations, the formal replacement $\nabla Y^2 \to \nabla Y^{\diamond 2}$ corresponds to the "subtraction of ∞" in the limit $\varepsilon \to 0$. From this perspective, instead of (8.14), we actually rather solve

$$\imath\partial_t u = \Delta u + u(\xi - \infty) + \lambda|u|^{2\sigma}u,\ u(0) = u_0\,, \tag{8.16}$$

where "$\infty = \lim_{\varepsilon\to 0}\mathbb{E}[|\nabla Y_\varepsilon|^2]$". This is almost the same situation we found in Chapter 4 where we had to subtract a diverging constant from the (discrete) approximation to the parabolic Anderson model and obtained a renormalized equation in the limit (Corollary 4.3.5).

We will read (8.15) as a rigorous formulation of (8.16) and consider solely this equation from now on. We assume, as in [DW16], that the initial condition u_0 is "controlled by Y" in the sense $v_0 = u_0 e^Y \in H^2(\mathbb{R}^2, \langle x\rangle^{\delta_0})$ for some

$$\delta_0 \in (0, 1/2)\,,$$

which we fix from now on. The solution v to (8.15) has then, at least formally, the conserved quantities

$$\begin{aligned}\tilde{N}(v) &= \int_{\mathbb{R}^2} |v|^2 e^{-2Y}\,\mathrm{d}x\,,\\ \tilde{H}(v) &= \int_{\mathbb{R}^2}\left(\frac{1}{2}|\nabla v|^2 - \frac{1}{2}|v|^2\widetilde{\nabla Y^{\diamond 2}} - \frac{\lambda}{2\sigma+2}|v|^{2\sigma+2}e^{-2\sigma Y}\right)e^{-2Y}\,\mathrm{d}x\,,\end{aligned} \tag{8.17}$$

where, as above, we evaluate $v = v(t)$ at some point $t \in [0, T]$ which we suppressed together with the dependency on x in our notation and where we introduced the notation $\widetilde{\nabla Y^{\diamond 2}} := \nabla Y^{\diamond 2} - \varphi * \xi$.

Our aim is to solve (8.15) by approximation via a smoothened equation

$$\imath\partial_t v_\varepsilon = \Delta v_\varepsilon + v_\varepsilon\widetilde{\nabla Y_\varepsilon^{\diamond 2}} - 2\nabla v_\varepsilon\cdot\nabla Y_\varepsilon + \lambda|v_\varepsilon e^{-Y_\varepsilon}|^{2\sigma}v_\varepsilon,\; v_\varepsilon(0) = v_0\,, \tag{8.18}$$

where Y_ε is defined as above via the mollification $Y_\varepsilon = \rho_\varepsilon * Y$ and with $\widetilde{\nabla Y_\varepsilon^{\diamond 2}} := \nabla Y_\varepsilon^{\diamond 2} - \varphi * \xi_\varepsilon$. Equation (8.18) has a (unique) solution for any $T > 0$ in

$$C([0,T]; H^2(\mathbb{R}^2, \langle x\rangle^{-\delta})) \cap C([0,T]; H^\gamma(\mathbb{R}^2, \langle x\rangle^{\delta'}))$$

for any $\delta > 0$, $\gamma \in (1, 2)$ and $\delta' < (1 - \frac{\gamma}{2})\delta_0$, see [Caz03, Section 3.6][1].

(8.18) has the conserved quantities

$$\begin{aligned}\tilde{N}(v_\varepsilon) &= \int_{\mathbb{R}^2} |v_\varepsilon|^2 e^{-2Y_\varepsilon}\,\mathrm{d}x\\ \tilde{H}_\varepsilon(v_\varepsilon) &= \int_{\mathbb{R}^2}\left(\frac{1}{2}|\nabla v_\varepsilon|^2 - \frac{1}{2}|v_\varepsilon|^2\widetilde{\nabla Y_\varepsilon^{\diamond 2}} - \frac{\lambda}{2\sigma+2}|v_\varepsilon|^{2\sigma+2}e^{-2\sigma Y_\varepsilon}\right)e^{-2Y_\varepsilon}\,\mathrm{d}x\,.\end{aligned}$$

which are well-defined due to $v_\varepsilon \in C([0,T]; H^\gamma(\mathbb{R}^2, \langle x\rangle^{\delta}))$ for $\gamma \in (1, 2)$ and $\delta < (1 - \frac{\gamma}{2})\delta_0$.

[1] The growth result is not contained in [Caz03] but follows from the same arguments than below if one first cuts-off the potential ξ_ε, then derives bounds independent of the truncation and finally removes the latter.

8.3 Moments and a priori bound in H^1

We start by a small lemma that allows us to control moments of v_ε by its derivatives.

Lemma 8.3.1. *Let v_ε be the unique solution to* (8.18) *on* $[0, T]$. *We then have for* $\delta \in (0, \delta_0)$ *and* $\delta' < 1 - 2\delta$

$$\sup_{t \in [0,T]} \int_{\mathbb{R}^2} |\langle x \rangle^\delta v_\varepsilon|^2 \, \mathrm{d}x \leqslant K_\varepsilon \int_{\mathbb{R}^2} |\langle x \rangle^{\delta_0} v_0|^2 \, \mathrm{d}x + T K_\varepsilon \sqrt{\tilde{N}(v_0)} \, \|\nabla v_\varepsilon\|_{C([0,T]; L^2(\mathbb{R}^2, \langle x \rangle^{-\delta'}))} \, .$$

Proof. We have, using that v_ε solves (8.18) and integrating by parts,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2} |\langle x \rangle^\delta \, v_\varepsilon|^2 e^{-2Y_\varepsilon} \, \mathrm{d}x &= 2 \operatorname{Re} \left(\int_{\mathbb{R}^2} \langle x \rangle^{2\delta} \, \partial_t v_\varepsilon \cdot \overline{v}_\varepsilon \, e^{-2Y_\varepsilon} \, \mathrm{d}x \right) \\ &= 2 \operatorname{Im} \left(\int_{\mathbb{R}^2} \langle x \rangle^{2\delta} (\Delta v_\varepsilon - 2 \nabla v_\varepsilon \cdot \nabla Y_\varepsilon) \overline{v}_\varepsilon \, e^{-2Y_\varepsilon} \, \mathrm{d}x \right) \\ &= -2 \operatorname{Im} \left(\int_{\mathbb{R}^2} \nabla \langle x \rangle^{2\delta} \cdot \nabla v_\varepsilon \, \overline{v}_\varepsilon e^{-2Y_\varepsilon} \, \mathrm{d}x \right) \\ &\leqslant C \int_{\mathbb{R}^2} \langle x \rangle^{2\delta - 1} |\nabla v_\varepsilon| \, |v_\varepsilon| \, e^{-2Y_\varepsilon} \, \mathrm{d}x \, . \end{aligned}$$

for some (deterministic) constant $C > 0$. Integrating over $[0, T]$ then shows

$$\begin{aligned} \sup_{t \in [0,T]} \int_{\mathbb{R}^2} |\langle x \rangle^\delta v_\varepsilon|^2 e^{-2Y_\varepsilon} \, \mathrm{d}x \leqslant & \int_{\mathbb{R}^2} |\langle x \rangle^\delta v_0|^2 e^{-2Y_\varepsilon} \\ & + CT \sup_{t \in [0,T]} \int_{\mathbb{R}^2} \langle x \rangle^{2\delta - 1} |\nabla v_\varepsilon| \, |v_\varepsilon| \, e^{-2Y_\varepsilon} \, \mathrm{d}x \, , \end{aligned}$$

so that the desired estimate follows with the Cauchy-Schwarz inequality and (8.8). □

We now derive an H^1 bound for v_ε. This is essentially based on an energy estimate, similar as in [DW16], but using Lemma 8.3.1 to control appearing moments.

Proposition 8.3.2. *Let v_ε be the unique solution of* (8.18) *with $\lambda \leqslant 0$ or $\sigma < 1$ on* $[0, T]$, *we then have for any $\delta > 0$*

$$\|v_\varepsilon\|_{C([0,T]; H^1(\mathbb{R}^2, \langle x \rangle^{-\delta}))} \leqslant K_\varepsilon (1 + \|v_0\|^a_{H^1(\mathbb{R}^2, \langle x \rangle^{\delta_0})}) \, ,$$

for some deterministic $a > 0$.

Proof. Note first that $\|v_\varepsilon\|_{C([0,T];L^2(\mathbb{R}^2,\langle x\rangle^{-\delta}))} \leqslant K_\varepsilon \|v_0\|^2_{L^2(\mathbb{R}^2,\langle x\rangle^{\delta_0})}$ is clear by conservation of mass and (8.8). Observe further that the claim follows if we can prove it for an arbitrarily small $\delta > 0$.

By the conservation of energy we obtain

$$\int_{\mathbb{R}^2} |\nabla v_\varepsilon|^2 e^{-2Y_\varepsilon} \mathrm{d}x = 2\tilde{H}_\varepsilon(v_0) + \int_{\mathbb{R}^2} \left(|v_\varepsilon|^2 \, \widetilde{\nabla Y_\varepsilon^{\diamond 2}} + \frac{\lambda}{\sigma+1} |v_\varepsilon|^{2\sigma+2} e^{-2\sigma Y_\varepsilon} \right) e^{-2Y_\varepsilon} \, \mathrm{d}x \,. \tag{8.19}$$

We estimate the first part of the integral on the right hand side by duality and Besov multiplication rules, that is *(iii)* and *(iv)* of Lemma 8.1.1,

$$\begin{aligned} \int_{\mathbb{R}^2} e^{-2Y_\varepsilon} |v_\varepsilon|^2 \widetilde{\nabla Y_\varepsilon^{\diamond 2}} &\overset{(iii)}{\lesssim} \|e^{-2Y_\varepsilon}|v_\varepsilon|^2\|_{\mathcal{B}_1^{\frac{1}{2}-2\kappa}(\mathbb{R}^2,\langle x\rangle^{\delta'})} \cdot \|\widetilde{\nabla Y_\varepsilon^{\diamond 2}}\|_{\mathcal{C}^{-\frac{1}{2}+2\kappa}(\mathbb{R}^2,\langle x\rangle^{-\delta'})} \\ &\overset{(iv)\,\&\,(8.8)}{\leqslant} K_\varepsilon \cdot \||v_\varepsilon|^2\|_{\mathcal{B}_1^{\frac{1}{2}-\kappa}(\mathbb{R}^2,\langle x\rangle^{\delta})} \overset{(iv)}{\leqslant} K_\varepsilon \, \|v_\varepsilon\|^2_{H^{\frac{1}{2}}(\mathbb{R}^2,\langle x\rangle^{\delta/2})} \,, \end{aligned}$$

where we took some (arbitrary) $\delta' \in (0,\delta)$, $\kappa \in (0,1/4)$ for the application of the multiplication rule in Lemma 8.1.1 and to bound e^{-2Y} via (8.8). Now, using weighted interpolation (*(v)* of Lemma 8.1.1) and Lemma 8.3.1 we have for δ small enough such that $2\delta < \delta_0$ and $\delta < 1 - 2\cdot 2\delta$

$$\begin{aligned} \|v_\varepsilon\|^2_{H^{\frac{1}{2}}(\mathbb{R}^2,\langle x\rangle^{\delta/2})} &\leqslant \|v_\varepsilon\|_{L^2(\mathbb{R}^2,\langle x\rangle^{2\delta})} \|v_\varepsilon\|_{H^1(\mathbb{R}^2,\langle x\rangle^{-\delta})} \\ &\leqslant K_\varepsilon \|v_0\|_{L^2(\mathbb{R}^2,\langle x\rangle^{\delta_0})} (1 + \|v_\varepsilon\|^{3/2}_{H^1(\mathbb{R}^2,\langle x\rangle^{-\delta})}) \,. \end{aligned}$$

Putting this into (8.19) and applying the Young product inequality (8.7) yields

$$\|v_\varepsilon\|^2_{H^1(\mathbb{R}^2,\langle x\rangle^{-\delta})} \lesssim \tilde{H}_\varepsilon(v_0) + K_\varepsilon(1 + \|v_0\|^a_{L^2(\mathbb{R}^2,\langle x\rangle^{\delta_0})}) + \lambda \int_{\mathbb{R}^2} |v_\varepsilon|^{2\sigma+2} e^{-(2\sigma+2)Y_\varepsilon} \, \mathrm{d}x \,. \tag{8.20}$$

If $\lambda \leqslant 0$ the last term is non-positive and can be dropped, otherwise consider the case $\sigma < 1$. Choose in the following $\kappa > 0$ so small that $\sigma + \kappa/2 < 1$. Fix further $\overline{\delta} \in (0,\delta_0)$, $\overline{\delta}' > 0$ such that $\kappa\overline{\delta} - (1-\kappa)\overline{\delta}' > 0$ and pick finally $\delta > 0$ so small that we have both $\frac{\sigma}{\sigma+1}(-\delta) + \frac{1}{\sigma+1}(\kappa\overline{\delta} - (1-\kappa)\overline{\delta}') > 0$ and $\delta < 1 - 2\overline{\delta}$. We have by the Sobolev embedding (*(ii)* in Lemma 8.1.1)

$$H^{\frac{\sigma}{\sigma+1}}\big(\mathbb{R}^2, \langle x\rangle^{\frac{\sigma}{\sigma+1}(-\delta) + \frac{1}{\sigma+1}(\kappa\overline{\delta} - (1-\kappa)\overline{\delta}')}\big) \subseteq L^{2\sigma+2}\big(\mathbb{R}^2, \langle x\rangle^{\frac{\sigma}{\sigma+1}(-\delta) + \frac{1}{\sigma+1}(\kappa\overline{\delta} - (1-\kappa)\overline{\delta}')}\big)$$

since the Sobolev number of both spaces is equal:

$$\frac{\sigma}{\sigma+1} - \frac{2}{2} = -\frac{1}{\sigma+1} = -\frac{2}{2\sigma+2}.$$

Using this continuous embedding, weighted interpolation (*(v)* of Lemma 8.1.1), (8.8), Lemma 8.3.1 and conservation of mass we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} |v_\varepsilon|^{2\sigma+2} e^{-(2\sigma+2)Y_\varepsilon}\, \mathrm{d}x &\overset{(8.8)}{\leqslant} K_\varepsilon \|v_\varepsilon\|^{2\sigma+2}_{H^{\frac{\sigma}{\sigma+1}}\left(\mathbb{R}^2, \langle x\rangle^{\frac{\sigma}{\sigma+1}(-\delta)+\frac{1}{\sigma+1}(\kappa\bar{\delta}-(1-\kappa)\bar{\delta}')}\right)} \\
&\overset{\text{Lem. 8.1.1 } (v)}{\leqslant} K_\varepsilon \|v_\varepsilon\|^{(2\sigma+2)\frac{\sigma}{\sigma+1}}_{H^1(\mathbb{R}^2,\langle x\rangle^{-\delta})} \|v_\varepsilon\|^{(2\sigma+2)(1-\frac{\sigma}{\sigma+1})}_{L^2(\mathbb{R}^2,\langle x\rangle^{\kappa\bar{\delta}-(1-\kappa)\bar{\delta}'})} \\
&= K_\varepsilon \|v_\varepsilon\|^{2\sigma}_{H^1(\mathbb{R}^2,\langle x\rangle^{-\delta})} \|v_\varepsilon\|^{2}_{L^2(\mathbb{R}^2,\langle x\rangle^{\kappa\bar{\delta}-(1-\kappa)\bar{\delta}'})} \\
&\overset{\text{Lem. 8.1.1 } (v)}{\leqslant} K_\varepsilon \|v_\varepsilon\|^{2\sigma}_{H^1(\mathbb{R}^2,\langle x\rangle^{-\delta})} \|v_\varepsilon\|^{2\kappa}_{L^2(\mathbb{R}^2,\langle x\rangle^{\bar{\delta}})} \|v_\varepsilon\|^{2(1-\kappa)}_{L^2(\mathbb{R}^2,\langle x\rangle^{-\bar{\delta}'})} \\
&\overset{\text{Lem. 8.3.1}}{\leqslant} K_\varepsilon \big(1+\|v_0\|^a_{L^2(\mathbb{R}^2,\langle x\rangle^{\delta_0})}\big)\big(1+\|v_\varepsilon\|^{2(\sigma+\kappa/2)}_{H^1(\mathbb{R}^2,\langle x\rangle^{-\delta})}\big)\,.
\end{aligned}$$

Together with (8.20) we get, by a further application of the Young product inequality (8.7), the estimate

$$\|v_\varepsilon\|_{C([0,T];H^1(\mathbb{R}^2,\langle x\rangle^{-\delta}))} \lesssim \tilde{H}_\varepsilon(v_0) + K_\varepsilon \left(1+\|v_0\|^a_{L^2(\mathbb{R}^2,\langle x\rangle^{\delta_0})}\right),$$

which implies the desired inequality. □

Combining Proposition 8.3.2 and Lemma 8.3.1 gives a uniform bound on the moments of v_ε.

Corollary 8.3.3. *In the setup of Proposition 8.3.2 we have for $\gamma \in [0,1)$ and $\delta < (1-\gamma)\delta_0$*

$$\|v_\varepsilon\|_{C([0,T];H^\gamma(\mathbb{R}^2,\langle x\rangle^{\delta}))} \leqslant K_\varepsilon\big(1+\|v_0\|^a_{H^1(\mathbb{R}^2,\langle x\rangle^{\delta_0})}\big)\,,$$

for some deterministic $a > 0$.

Proof. Inserting the estimate of Proposition 8.3.2 in Lemma 8.3.1 we obtain for $\delta < \delta_0$

$$\|v_\varepsilon\|_{C([0,T];L^2(\mathbb{R}^2,\langle x\rangle^{\delta}))} \leqslant K_\varepsilon\big(1+\|v_0\|^a_{H^1(\mathbb{R}^2,\langle x\rangle^{\delta_0})}\big)\,.$$

The result then follows by applying the weighted Besov interpolation (*(v)* of Lemma 8.1.1) together with Proposition 8.3.2. □

8.4 Local existence

Although the bound from Corollary 8.3.3 is enough to define the limit of $v_\varepsilon \widetilde{\nabla Y_\varepsilon^{\diamond 2}}$ in equation (8.18), we need a bound in H^γ for $\gamma > 1$ to control the product

$$-2\nabla v_\varepsilon \cdot \nabla Y_\varepsilon\,.$$

In [DW16] this was achieved by estimating the L^2 norm of the time derivative $w_\varepsilon = \frac{\mathrm{d}}{\mathrm{d}t} v_\varepsilon$, which morally corresponds to bounding the H^2 bound of v_ε. A key role in their estimate was played by the Brezis-Gallouet inequality from Lemma 8.1.2 which needs a bound in the critical Sobolev space with Sobolev number 0, that is H^1. Although we were able to derive a bound in this space in Proposition 8.3.2 above, we can so far only control the *decay* of v_ε in H^γ for $\gamma < 1$ via Corollary 8.3.3 so that we cannot simply apply the Brezis-Gallouet inequality at this point.

We first prove instead that we can bound the H^γ norm of v_ε with $\gamma > 1$ provided one has a bound in L^∞. Using then that $L^\infty \subseteq H^\gamma$ for $\gamma > 1$ we can conclude local existence in Theorem 8.4.4 below. In Section 8.5 we show that the Brezis-Gallouet inequality from Lemma 8.1.2 can be modified to still yield global existence provided $\sigma \in (0, 1/2)$. This result is stated in Theorem 8.5.2.

Lemma 8.4.1. *Let v_ε be the unique solution to* (8.18) *with $\lambda \leqslant 0$ or $\sigma < 1$ on $[0, T]$. We then have for $\delta > 0$*

$$\begin{aligned}\|v_\varepsilon\|_{C([0,T];H^2(\mathbb{R}^2,\langle x\rangle^{-\delta}))} &\leqslant K_\varepsilon\big(1 + \|v_0\|^a_{H^2(\mathbb{R}^2,\langle x\rangle^{\delta_0})}\big)\\ &\quad\times e^{CT\|v_\varepsilon e^{-Y_\varepsilon}\|^{2\sigma}_{C([0,T];L^\infty(\mathbb{R}^2))}}\big(1 + |\log(\varepsilon)|^a\big)\,,\end{aligned}$$

for some deterministic constants $a, C > 0$.

Proof. We consider as in [DW16] the quantity $w_\varepsilon = \partial_t v_\varepsilon$ which satisfies the equation

$$\begin{aligned}\imath\partial_t w_\varepsilon &= \Delta w_\varepsilon + w_\varepsilon\,\widetilde{\nabla Y_\varepsilon^{\diamond 2}} - 2\nabla w_\varepsilon \cdot \nabla Y_\varepsilon + \lambda|v_\varepsilon e^{-Y_\varepsilon}|^{2\sigma} w_\varepsilon\\ &\quad + \sigma\lambda v_\varepsilon|v_\varepsilon|^{2\sigma-2}\, 2\,\mathrm{Re}\,(w_\varepsilon \overline{v}_\varepsilon)\; e^{-2\sigma Y_\varepsilon}\end{aligned}$$

and whose mass evolves like

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} |w_\varepsilon|^2 e^{-2Y_\varepsilon}\,\mathrm{d}x &= 2\sigma\lambda\int_{\mathbb{R}^2} \mathrm{Im}\,(\overline{w}_\varepsilon v_\varepsilon)\,\mathrm{Re}\,(w_\varepsilon\overline{v}_\varepsilon)\; |v_\varepsilon|^{2\sigma-2} e^{-(2\sigma+2)Y_\varepsilon}\mathrm{d}x\\ &\leqslant C\|v_\varepsilon e^{-Y_\varepsilon}\|^{2\sigma}_{L^\infty(\mathbb{R}^2)}\int_{\mathbb{R}^2}|w_\varepsilon|^2 e^{-2Y_\varepsilon}\,\mathrm{d}x\,.\end{aligned} \tag{8.21}$$

for some deterministic constant $C > 0$. Gronwall's lemma provides then

$$\int_{\mathbb{R}^2} |w_\varepsilon(t)|^2 e^{-2Y_\varepsilon} \leqslant \int_{\mathbb{R}^2} |w(0)|^2 e^{-2Y_\varepsilon}\cdot e^{CT\|v_\varepsilon e^{-Y_\varepsilon}\|^{2\sigma}_{C([0,T];L^\infty(\mathbb{R}^2))}}\,. \tag{8.22}$$

Recall that $\imath w_\varepsilon = \Delta v_\varepsilon + v_\varepsilon\,\widetilde{\nabla Y_\varepsilon^{\diamond 2}} - 2\nabla v_\varepsilon \cdot \nabla Y_\varepsilon + \lambda|v_\varepsilon|^{2\sigma} v_\varepsilon e^{-2\sigma Y_\varepsilon}$. By Sobolev embedding (*(ii)* in Lemma 8.1.1) we have $\|v_0\|_{L^q(\mathbb{R}^2,\langle x\rangle^\delta)}, \|\nabla v_0\|_{L^q(\mathbb{R}^2,\langle x\rangle^\delta)} \lesssim \|v_0\|_{H^2(\mathbb{R}^2,\langle x\rangle^\delta)}$ for $q \in [2,\infty)$, $\delta \leqslant \delta_0$. Choose $q > 2$ but close enough to 2 such that q' with $\frac{1}{2} = \frac{1}{q} + \frac{1}{q'}$

satisfies $q' \cdot \delta_0 > 4$. We then have with Lemma 8.1.7

$$\begin{aligned}
&\|w_\varepsilon(0)\|_{L^2(\mathbb{R}^2,\langle x\rangle^{\frac{\delta_0}{2}})} \lesssim \|v_\varepsilon(0)\|_{H^2(\mathbb{R}^2,\langle x\rangle^{\frac{\delta_0}{2}})} + \|v_\varepsilon(0)\|_{L^q(\mathbb{R}^2,\langle x\rangle^{\delta_0/2})}\|\widetilde{\nabla Y_\varepsilon^{\diamond 2}}\|_{L^{q'}(\mathbb{R}^2,\langle x\rangle^{-\delta_0/2})} \\
&+ 2\|\nabla v_\varepsilon(0)\|_{L^q(\mathbb{R}^2,\langle x\rangle^{\delta_0})}\|\nabla Y_\varepsilon\|_{L^{q'}(\mathbb{R}^2,\langle x\rangle^{-\delta_0/2})} \\
&+ \lambda\||v_\varepsilon(0)|^{2\sigma+1}\|_{L^q(\mathbb{R}^2,\langle x\rangle^{\delta_0})}\|e^{-2\sigma Y_\varepsilon}\|_{L^{q'}(\mathbb{R}^2,\langle x\rangle^{-\frac{\delta_0}{2}})} \\
&\leqslant K_\varepsilon(1+\|v_0\|^a_{H^2(\mathbb{R}^2,\langle x\rangle^{\delta_0})})\cdot(1+|\log\varepsilon|^a)\,,
\end{aligned} \tag{8.23}$$

where we used Corollary 8.3.3, Sobolev embedding (*(ii)* of Lemma 8.1.1) and $v_\varepsilon(0) = v_0$ in the last step. On the other hand we have for $\delta > 0$ and

$$\begin{aligned}
\|\Delta v_\varepsilon(t)\|_{L^2(\mathbb{R}^2,\langle x\rangle^{-\delta})} &\lesssim \|w_\varepsilon(t)\|_{L^2(\mathbb{R}^2,\langle x\rangle^{-\delta})} + \|v_\varepsilon(t)\|_{L^q(\mathbb{R}^2,\langle x\rangle^{-3\delta/4})}\|\widetilde{\nabla Y_\varepsilon^{\diamond 2}}\|_{L^{q'}(\mathbb{R}^2,\langle x\rangle^{-\delta/4})} \\
&+ 2\|\nabla v_\varepsilon(t)\|_{L^q(\mathbb{R}^2,\langle x\rangle^{-3\delta/4})}\|\nabla Y_\varepsilon\|_{L^{q'}(\mathbb{R}^2,\langle x\rangle^{-\delta/4})} \\
&+ \||v_\varepsilon(t)|^{2\sigma+1}\|_{L^q(\mathbb{R}^2,\langle x\rangle^{-3\delta/4})}\|e^{-(2\sigma+1)Y_\varepsilon}\|_{L^{q'}(\mathbb{R}^2,\langle x\rangle^{-\delta/4})}\,,
\end{aligned}$$

where now $q \in (2,4)$ is small enough such that q' with $\frac{1}{2} = \frac{1}{q} + \frac{1}{q'}$ satisfies $q' \cdot \delta > 8$. By Sobolev embedding and interpolation we have

$$\begin{aligned}
\|v_\varepsilon(t)\|_{L^q(\mathbb{R}^2,\langle x\rangle^{-3\delta/4})}, \|\nabla v_\varepsilon(t)\|_{L^q(\mathbb{R}^2,\langle x\rangle^{-3\delta/4})}\| &\lesssim \|v_\varepsilon(t)\|_{H^{3/2}(\mathbb{R}^2,\langle x\rangle^{-3\delta/4})} \\
&\leqslant \|v_\varepsilon(t)\|^{\frac{1}{2}}_{H^1(\mathbb{R}^2,\langle x\rangle^{-\delta/2})}\|v_\varepsilon(t)\|^{\frac{1}{2}}_{H^2(\mathbb{R}^2,\langle x\rangle^{-\delta})}\,.
\end{aligned}$$

Applying Proposition 8.3.2 we therefore obtain for some $a > 0$

$$\begin{aligned}
\|\Delta v_\varepsilon(t)\|_{L^2(\mathbb{R}^2,\langle x\rangle^{-\delta})} &\lesssim \|w_\varepsilon(t)\|_{L^2(\mathbb{R}^2,\langle x\rangle^{-\delta})} \\
&+ K_\varepsilon(1+\|v_0\|^a_{H^2(\mathbb{R}^2,\langle x\rangle^{\delta_0})})(1+|\log\varepsilon|^a)\,\|v_\varepsilon(t)\|^{1/2}_{H^2(\mathbb{R}^2,\langle x\rangle^{-\delta})}\,.
\end{aligned}$$

It is easy to see $\|g\|_{H^2(\mathbb{R}^2,\langle x\rangle^{-\delta})} \lesssim \|g\|_{H^1(\mathbb{R}^2,\langle x\rangle^{-\delta})} + \|\Delta g\|_{L^2(\mathbb{R}^2,\langle x\rangle^{-\delta})}$, via (8.2) and the unweighted analogue of this estimate, so that we obtain

$$\|v_\varepsilon(t)\|_{H^2(\mathbb{R}^2,\langle x\rangle^{-\delta})} \lesssim \|w_\varepsilon(t)\|_{L^2(\mathbb{R}^2,\langle x\rangle^{-\delta})} + K_\varepsilon(1+\|v_0\|^a_{H^2(\mathbb{R}^2,\langle x\rangle^{\delta_0})})(1+|\log\varepsilon|^a)\,. \tag{8.24}$$

Applying (8.24) to the left hand side and (8.23) to the right hand side of (8.22) (using (8.8)) shows the desired estimate. □

Remark 8.4.2. *There was a technical subtlety in this proof which we hid from the reader for the sake of a clearer argument. Note that we do not know if the time derivative in* (8.21) *is well-defined. Due to the non-integer value of σ it is not clear that we have even for smooth initial conditions smooth solutions which would allow for such an operation. For a rigorous argument one really has to work instead with* *the solution* v_ε^n *to* $\imath\partial_t v_\varepsilon^n = \Delta v_\varepsilon^n + 2v_\varepsilon^n \widetilde{\nabla Y_\varepsilon^{\diamond 2}} - 2\nabla v_\varepsilon^n \cdot \nabla Y^\varepsilon + (|v_\varepsilon^n e^{-Y_\varepsilon}|^2 + \frac{1}{n})^\sigma$, *started in smooth* $v_\varepsilon^n(0) \in \mathcal{S}(\mathbb{R}^d)$ *that converge to* v_0 *for* $n \to \infty$. *One then easily derives bounds as above. Working with the back-transformed solution* $u_\varepsilon^n = v_\varepsilon^n e^{-Y_\varepsilon}$ *one can prove a* $L^\infty(\mathbb{R}^2)$ *bound, uniform in* n, *on* $v_n^\varepsilon e^{-Y_\varepsilon}$. *One thus gets boundedness, uniform in* n, *of* $\|v_\varepsilon^n\|_{H^2(\mathbb{R}^2,\langle x\rangle^{-\delta})}$. *By choice of a weakly convergent subsequence and the compact embedding* $H^\gamma(\mathbb{R}^2,\langle x\rangle^\delta) \subseteq H^{\gamma'}(\mathbb{R}^2,\langle x\rangle^{\delta'})$, $\gamma' < \gamma, \delta' < \delta$ *one concludes.*

This estimate is sufficient to prove local existence. We follow [DW16] in the consideration of the differences of the dyadic subsequence, which cancels the logarithmic factor in Lemma 8.4.1. Recall that the notation K_k in the following stands for a random constant of the form $K_k = K_{2^{-k}} K_{2^{-k-1}}$ that can be bounded almost surely in k. To derive the latter property we will always use the polynomial convergence rates from (8.11), (8.12) and (8.13) together with the Borell-Cantelli lemma so that for example for $\alpha \in (0,1)$ and $\delta > 0$

$$\|Y_{2^{-k}} - Y_{2^{-k-1}}\|_{\mathcal{C}^\alpha(\mathbb{R}^2,\langle x\rangle^{-\delta})} \leqslant \|Y_{2^{-k}} - Y\|_{\mathcal{C}^\alpha(\mathbb{R}^2,\langle x\rangle^{-\delta})} + \|Y - Y_{2^{-k-1}}\|_{\mathcal{C}^\alpha(\mathbb{R}^2,\langle x\rangle^{-\delta})} \leqslant K_k\,.$$

Lemma 8.4.3. *Let* $v_{2^{-k}}$ *be the unique solution to* (8.18) *on* $[0,T]$ *with* $\varepsilon = 2^{-k}$ *and* $\lambda \leqslant 0$ *or* $\sigma < 1$. *We then have for* $\gamma \in (0,2)$, $\delta < (1-\frac{\gamma}{2})\delta_0$

$$\|v_{2^{-k}} - v_{2^{-k-1}}\|_{C([0,T];H^\gamma(\mathbb{R}^2,\langle x\rangle^\delta))} \leqslant$$
$$K_k 2^{-k\kappa}(1 + \|v_0\|^a_{H^2(\mathbb{R}^2,\langle x\rangle^{\delta_0})})\, e^{CT(\|v_{2^{-k}} e^{-Y_{2^{-k}}}\|^{2\sigma}_{C([0,T];L^\infty(\mathbb{R}^2))} + \|v_{2^{-k-1}} e^{-Y_{2^{-k-1}}}\|^{2\sigma}_{C([0,T];L^\infty(\mathbb{R}^2))})}\,.$$

for some $\kappa > 0$ *and* $C, a > 0$, *where the sequence of random constants* K_k *is bounded almost surely.*

Proof. The difference $r_k = v_{2^{-k}} - v_{2^{-k-1}}$ satisfies the equation

$$\begin{aligned}\imath\partial_t r_k &= \Delta r_k + r_k\, \widetilde{\nabla Y^{\diamond 2}_{2^{-k-1}}} - 2\nabla r_k \cdot \nabla Y_{2^{-k-1}} + v_{2^{-k}}(\widetilde{\nabla Y^{\diamond 2}_{2^{-k-1}}} - \widetilde{\nabla Y^{\diamond 2}_{2^{-k}}})\\ &\quad - 2\nabla v_{2^{-k}} \cdot (\nabla Y_{2^{-k-1}} - \nabla Y_{2^{-k}})\\ &\quad + \lambda(|v_{2^{-k}} e^{-Y_{2^{-k}}}|^{2\sigma} v_{2^{-k}} - |v_{2^{-k-1}} e^{-Y_{2^{-k-1}}}|^{2\sigma} v_{2^{-k-1}})\,,\end{aligned}$$

so that the “mass” of r_k evolves according to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} |r_k|^2 e^{-2Y_{2^{-k-1}}} = \mathrm{Im}\Bigg\{\int_{\mathbb{R}^2}\Big(v_{2^{-k}}(\widetilde{\nabla Y^{\diamond 2}_{2^{-k-1}}} - \widetilde{\nabla Y^{\diamond 2}_{2^{-k}}})\overline{r}_k e^{-2Y_{2^{-k-1}}}\\ &- 2\nabla v_{2^{-k}} \cdot (\nabla Y_{2^{-k-1}} - \nabla Y_{2^{-k}})\overline{r}_k e^{-2Y_{2^{-k-1}}}\\ &+ \lambda(|v_{2^{-k}} e^{-Y_{2^{-k}}}|^{2\sigma} v_{2^{-k}} - |v_{2^{-k-1}} e^{-Y_{2^{-k-1}}}|^{2\sigma} v_{2^{-k-1}})\overline{r}_k e^{-2Y_{2^{-k-1}}}\Big)\,\mathrm{d}x\Bigg\}\,.\end{aligned}$$

Via *(iii)*, *(iv)* of Lemma 8.1.1 (Duality and Multiplication bound) and using interpolation between the bound in Corollary 8.3.3 and Lemma 8.4.1 we can estimate the first two terms on the right hand side, up to a constant, with an arbitrary $\varepsilon' \in (0, 1/2)$ by

$$\begin{aligned}
&\|v_{2^{-k}}\overline{r}_k e^{-2Y_{2^{-k-1}}}\|_{\mathcal{B}_1^{\frac{1}{2}-\varepsilon'}(\mathbb{R}^2,\langle x\rangle^{\delta'})} \|\widetilde{\nabla Y^{\diamond 2}_{2^{-k-1}}} - \widetilde{\nabla Y^{\diamond 2}_{2^{-k}}}\|_{\mathcal{C}^{-\frac{1}{2}+\varepsilon'}(\mathbb{R}^2,\langle x\rangle^{-\delta'})}\\
&+ \|\nabla v_{2^{-k}}\overline{r}_k e^{-2Y_{2^{-k-1}}}\|_{\mathcal{B}_1^{\frac{1}{2}-\varepsilon'}(\mathbb{R}^2,\langle x\rangle^{\delta'})} \|(\nabla Y_{2^{-k-1}} - \nabla Y_{2^{-k}})\|_{\mathcal{C}^{-\frac{1}{2}+\varepsilon'}(\mathbb{R}^2,\langle x\rangle^{-\delta'})}\\
&\leqslant K_k(\|v_{2^{-k}}\|^2_{H^{\frac{3}{2}}(\mathbb{R}^2,\langle x\rangle^{\delta'})} + \|v_{2^{-k-1}}\|^2_{H^{\frac{3}{2}}(\mathbb{R}^2,\langle x\rangle^{\delta'})})\, 2^{-k\kappa'}\\
&\leqslant (1 + \|v_0\|^a_{H^2(\mathbb{R}^2,\langle x\rangle^{\delta_0})})K_k 2^{-k\kappa'/2}\, e^{CT(\|v_{2^{-k}}\|^{2\sigma}_{C([0,T];L^\infty(\mathbb{R}^2))}+\|v_{2^{-k-1}}\|^{2\sigma}_{C([0,T];L^\infty(\mathbb{R}^2))})}\,,
\end{aligned}$$

for $\delta' \in (0, \delta_0/4)$ and $\kappa' < 1/2$. Up to a term $K_k 2^{-k\kappa}(1 + \|v_0\|^a_{H^1(\mathbb{R}^2,\langle x\rangle^{\delta_0})})$ we can reshape the third term as

$$\begin{aligned}
\mathrm{Im}\Big\{ \int_{\mathbb{R}^2} &\lambda(|v_{2^{-k}}e^{-Y_{2^{-k}}}|^{2\sigma} v_{2^{-k}}e^{-Y_{2^{-k}}}\\
&- |v_{2^{-k-1}}e^{-Y_{2^{-k-1}}}|^{2\sigma} v_{2^{-k-1}}e^{-Y_{2^{-k-1}}})\overline{r}_k e^{-Y_{2^{-k-1}}}\,\mathrm{d}x\Big\}
\end{aligned}$$

Recall that for $x, y \in \mathbb{C}$ $||x|^{2\sigma}x - |y|^{2\sigma}y| \leqslant C(|x|^{2\sigma} + |y|^{2\sigma})\,|x - y|$ (see for example [Caz03, p. 86]) so that we obtain the upper bound

$$\int_{\mathbb{R}^2} (|v_{2^{-k}}e^{-Y_{2^{-k}}}|^{2\sigma} + |v_{2^{-k-1}}e^{-Y_{2^{-k-1}}}|^{2\sigma})|v_{2^{-k}}e^{-Y_{2^{-k}}} - v_{2^{-k-1}}e^{-Y_{2^{-k-1}}}|\overline{r}_k e^{-Y_{2^{-k-1}}}\,\mathrm{d}x\,,$$

Applying (8.12) and Corollary 8.3.3 we can bound this up to a term
$K_k(1 + \|v_0\|^a_{H^1(\mathbb{R}^2,\langle x\rangle^{\delta_0})})2^{-k\kappa}$ by

$$\begin{aligned}
&\int_{\mathbb{R}^2} (|v_{2^{-k}}e^{-Y_{2^{-k}}}|^{2\sigma} + |v_{2^{-k-1}}e^{-Y_{2^{-k-1}}}|^{2\sigma})|r_k|^2 e^{-2Y_{2^{-k-1}}}\,\mathrm{d}x\\
&\leqslant (\|v_{2^{-k}}e^{-Y_{2^{-k}}}\|^{2\sigma}_{L^\infty(\mathbb{R}^2)} + \|v_{2^{-k-1}}e^{-Y_{2^{-k-1}}}\|^{2\sigma}_{L^\infty(\mathbb{R}^2)}) \int_{\mathbb{R}^2} |r_k|^2 e^{-2Y_{2^{-k-1}}}\,\mathrm{d}x\,.
\end{aligned}$$

Putting everything together we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} |r_k|^2 e^{-2Y_{2^{-k-1}}} &\leqslant K_k 2^{-k\kappa}\, e^{CT(\|v_{2^{-k}}\|^{2\sigma}_{C([0,T];L^\infty)}+\|v_{2^{-k-1}}\|^{2\sigma}_{C([0,T];L^\infty(\mathbb{R}^2))})}\\
&+ C(\|v_{2^{-k}}e^{-Y_{2^{-k}}}\|^{2\sigma}_{L^\infty(\mathbb{R}^2)} + \|v_{2^{-k-1}}e^{-Y_{2^{-k-1}}}\|^{2\sigma}_{L^\infty}) \int_{\mathbb{R}^2} |r_k|^2 e^{-2Y_{2^{-k-1}}}\,\mathrm{d}x\,,
\end{aligned}$$

for some $\kappa, C > 0$. Application of Gronwall's lemma gives, together with (8.12) and Borell-Cantelli,

$$\begin{aligned}
&\|v_{2^{-k}} - v_{2^{-k-1}}\|_{C([0,T];L^2(\mathbb{R}^2,\langle x\rangle^{-\delta}))} \leqslant\\
&K_k 2^{-k\kappa}(1 + \|v_0\|_{H^2(\mathbb{R}^2,\langle x\rangle^{\delta_0})})\, e^{CT(\|v_{2^{-k}}e^{-Y_{2^{-k}}}\|^{2\sigma}_{C([0,T];L^\infty(\mathbb{R}^2))}+\|v_{2^{-k-1}}e^{-Y_{2^{-k-1}}}\|^{2\sigma}_{C([0,T];L^\infty(\mathbb{R}^2))})}\,.
\end{aligned}$$

for any $\delta > 0$. The desired estimate now follows by interpolation with Corollary 8.3.3 and Lemma 8.4.1. □

We are now in the position to prove local existence.

Theorem 8.4.4. *There is a (random) time $T > 0$ such that (8.15) with $\lambda \leqslant 0$ or $\sigma < 1$ has almost surely a (unique) solution v in $C([0,T]; H^\gamma(\mathbb{R}^2, \langle x\rangle^\delta))$ for $\gamma \in (1,2)$ and $\delta < (1-\frac{\gamma}{2})\delta_0$. The random variable $\|v_\varepsilon - v\|_{C([0,T];H^\gamma(\mathbb{R}^2,\langle x\rangle^\delta)}$ converges to 0 in probability.*

Proof. Let $M_T^N := \sup_{k\leqslant N} \|v_{2^{-k}}\|_{C([0,T];H^\gamma(\mathbb{R}^2,\langle x\rangle^\delta))}$, we obtain summing the estimate in Lemma 8.4.3

$$\begin{aligned} M_T^N &\leqslant \|v_1\|_{C([0,T];H^\gamma(\mathbb{R}^2,\langle x\rangle^\delta))} + \sum_{k=1}^{\infty} K_k 2^{-k\kappa}(1+\|v_0\|_{H^2(\mathbb{R}^2,\langle x\rangle^{\delta_0})})e^{CTM_T^N} \\ &\leqslant K(1+\|v_0\|_{C([0,T];H^2(\mathbb{R}^2,\langle x\rangle^\delta))})e^{CTM_T^N}, \end{aligned} \tag{8.25}$$

where we used $H^\gamma(\mathbb{R}^2,\langle x\rangle^\delta) \subseteq L^\infty(\mathbb{R}^2,\langle x\rangle^\delta)$ (which follows by combining Lemma 2.1.28 and *(i)* of Lemma 8.1.1) and where the random constant K is finite almost surely and moreover, by Minkowski's inequality, in any $L^p(\mathbb{P})$, $p \in [1,\infty)$. Using the time-continuity of $T \mapsto M_T^N$ for a fixed N, one gets then by the standard local estimate arguments from (8.25) the existence of a random time T, independent of N, such that for any $N \in \mathbb{N}$

$$M_T^N \leqslant 2K(1+\|v_0\|_{C([0,T];H^2(\mathbb{R}^2,\langle x\rangle^\delta))})$$

and thus we have that $\|v_{2^{-k}}\|_{C([0,T];H^\gamma(\mathbb{R}^2,\langle x\rangle^\delta))}$ is uniformly bounded. Reinserting this in Lemma 8.4.3 shows that $v_{2^{-k}}$ is a Cauchy sequence and we can conclude convergence to a v that solves (8.15).

To see uniqueness, suppose we are given two solutions $v,\, v' \in C([0,T]; H^\gamma(\mathbb{R}^2,\langle x\rangle^\delta))$. One then obtains for $h := v - v'$

$$\partial_t \|he^{-Y}\|^2_{L^2(\mathbb{R}^d)} = \lambda \mathrm{Im}\left(\int_{\mathbb{R}^2} (|v|^{2\sigma}v - |v'|^{2\sigma}v')\,\overline{h}e^{-(2\sigma+2)Y}\right) \lesssim_{v,v'} K\,\|he^{-Y}\|^2_{L^2(\mathbb{R}^d)}$$

where we applied once more $||v|^{2\sigma}v - |v'|^{2\sigma}v'| \lesssim (|v|^{2\sigma}+|v'|^{2\sigma})|h|$ from [Caz03, p. 86] together with $\|ve^{-Y}\|_{L^\infty(\mathbb{R}^d)} \lesssim K\,\|v\|_{C([0,T];H^\gamma(\mathbb{R}^2,\langle x\rangle^\delta))} \lesssim_v K$ (and similar for v') due to the embedding $H^\gamma \subseteq L^\infty$. Uniqueness then follows with Gronwall's inequality.

The convergence of probablity follows similar as in [DW16] by considering first $\|v_\varepsilon - v_{2^{-k}}\|_{C([0,T];H^\gamma(\mathbb{R}^2,\langle x\rangle^\delta))}$: Redoing then the proof of Lemma 8.4.3 but bounding $\|v_{2^{-k}}\|_{H^{3/2}(\mathbb{R}^2,\langle x\rangle^\delta)}$, $\|v_{2^{-k}}e^{-Y_{2^{-k}}}\|_{L^\infty(\mathbb{R}^2)}$ directly instead of applying Lemma 8.4.1 we can let $k \to \infty$ and the resulting estimate yields the convergence in probability. □

8.5 Global existence for $\sigma < 1/2$

In the case $\sigma < 1/2$ we now prove that there is a global solution to (8.15). As remarked at the beginning of Section 8.4 we cannot simply use the Brezis-Gallouet inequality as we have no uniform H^1 bound yet available that controls the decay of the solution v^ε to (8.18). However, we can reshape Lemma 8.1.2 to the following statement.

Lemma 8.5.1. *For the solution v_ε of* (8.18) *with $\lambda \leqslant 0$ or $\sigma < 1$ we have for $\gamma \in (1,2)$, $\kappa > 0$ and $\delta < (\gamma - 1)\delta_0$*

$$\begin{aligned} \|v_\varepsilon e^{-Y_\varepsilon}\|_{C([0,T];L^\infty(\mathbb{R}^2))} &\lesssim K_\varepsilon + (1 + |\log \varepsilon|^{1+\kappa})(1 + \|v_0\|^a_{H^1(\mathbb{R}^2,\langle x\rangle^{\delta_0})}) \\ &\quad + \log(1 + \|v_\varepsilon\|_{C([0,T];H^\gamma(\mathbb{R}^2,\langle x\rangle^{-\delta}))}) \,, \end{aligned}$$

for some deterministic $a > 0$. The sequence $(K_{2^{-k}})_{k\in\mathbb{N}}$ one obtains for for dyadic ε is bounded almost surely.

Proof. Choose $\delta_0' \in (0, \delta_0)$ and $\gamma' \in (1, \gamma)$ large enough so that we have $\delta < \frac{\gamma-\gamma'}{\gamma'}\delta_0'$ and thus $\delta' := \frac{\gamma-\gamma'}{\gamma}\delta_0' - \frac{\gamma'}{\gamma}\delta \in (0, \delta_0)$. Applying then the Brezis-Gallouet inequality from Lemma 8.1.2 we obtain with the Besov mulitplication rule, (8.8), Young's product inequality (8.7) and interpolation

$$\begin{aligned} \|v_\varepsilon e^{-Y_\varepsilon}\|_{L^\infty(\mathbb{R}^2)} &\lesssim (1 + \|v_\varepsilon e^{-Y_\varepsilon}\|_{H^1(\mathbb{R}^2)}) \sqrt{1 + \log(1 + \|v_\varepsilon e^{-Y_\varepsilon}\|_{\mathcal{C}^{\gamma-1}(\mathbb{R}^2)})} \\ &\lesssim (1 + \|v_\varepsilon e^{-Y_\varepsilon}\|_{H^1(\mathbb{R}^2)}) \sqrt{1 + \log(1 + K_\varepsilon \|v_\varepsilon\|_{\mathcal{C}^{\gamma'-1}(\mathbb{R}^2,\langle x\rangle^{\delta'})})} \\ &\leqslant K_\varepsilon + \|v_\varepsilon e^{-Y_\varepsilon}\|^2_{H^1(\mathbb{R}^2)} + \log(1 + \|v_\varepsilon\|_{H^{\gamma'}(\mathbb{R}^2,\langle x\rangle^{\delta'})}) \\ &\leqslant K_\varepsilon + \|v_\varepsilon e^{-Y_\varepsilon}\|^2_{H^1(\mathbb{R}^2)} + \log(1 + \|v_\varepsilon\|^{\frac{\gamma'}{\gamma}}_{H^\gamma(\mathbb{R}^2,\langle x\rangle^{-\delta})} \|v_\varepsilon|^{\frac{\gamma-\gamma'}{\gamma}}_{L^2(\mathbb{R}^2,\langle x\rangle^{\delta_0'})}) \,. \end{aligned}$$

Note that we have by the product rule

$$\|v_\varepsilon e^{-Y_\varepsilon}\|^2_{H^1(\mathbb{R}^2)} \lesssim \|v_\varepsilon e^{-Y_\varepsilon}\|^2_{L^2(\mathbb{R}^2)} + \|\nabla v_\varepsilon e^{-Y_\varepsilon}\|^2_{L^2(\mathbb{R}^2)} + \|v_\varepsilon e^{-Y_\varepsilon} \nabla Y_\varepsilon\|^2_{L^2(\mathbb{R}^2)} \,.$$

While the first term is bounded by conservation of mass, the second can be bounded by conservation of energy (8.19). For the last we apply Hölder's inequality and Lemma 8.1.7 to bound it by

$$\begin{aligned} \|v_\varepsilon e^{-Y_\varepsilon} \nabla Y_\varepsilon\|^2_{L^2(\mathbb{R}^2)} &\leqslant \|v_\varepsilon e^{-Y_\varepsilon}\|^2_{L^q(\mathbb{R}^2,\langle x\rangle^{\tilde\delta})} \|\nabla Y_\varepsilon\|^2_{L^{q'}(\mathbb{R}^2,\langle x\rangle^{-\tilde\delta})} \\ &\leqslant K_\varepsilon (1 + \|v_0\|^a_{H^1(\mathbb{R}^2,\langle x\rangle^{\delta_0})}) \, |\log \varepsilon|^{1+\kappa/2} \,, \end{aligned} \tag{8.26}$$

for some small $\tilde\delta > 0$ and $\frac{1}{2} = \frac{1}{q} + \frac{1}{q'}$ with $q' \in (2, \infty)$ large enough so that $\tilde\delta \cdot q' > 2$ and where we used Corollary 8.3.3 and 8.1.7 in the second step. Note we introduced an additional factor $|\log\varepsilon|^{\kappa/2}$ so that the constants K_ε are bounded almost surely for dyadic $\varepsilon = 2^{-k}$ by (8.11), (8.12) and Borel-Cantelli. Inserting (8.26) above shows the desired inequality. □

The combination of Lemma 8.5.1 with Lemma 8.4.1 and 8.4.3 gives global existence provided $\sigma < \frac{1}{2}$.

Theorem 8.5.2. *The equation* (8.15) *with $\lambda = 0$ or $\sigma \in (0, 1/2)$ has for every $T > 0$ a unique solution $v \in C([0,T]; H^\gamma(\mathbb{R}^2, \langle x\rangle^\delta))$ for $\gamma \in (1,2)$ and $\delta < (1 - \frac{\gamma}{2})\delta_0$. The solutions v_ε of* (8.18) *converge in probability in $C([0,T]; H^\gamma(\mathbb{R}^2, \langle x\rangle^\delta))$ to v.*

Proof. We assume $\sigma \in (0, 1/2)$, since the linear case $\lambda = 0$ is trivially included in this range. Inserting the Brezis-Gallouet inequality in Lemma 8.5.1 in the bound in Lemma 8.4.1 gives for some $a > 0$

$$\begin{aligned}\|v_{2^{-k}}\|_{H^\gamma(\mathbb{R}^2,\langle x\rangle^\delta)} &\lesssim \left(1 + \|v_0\|^a_{H^2(\mathbb{R}^2,\langle x\rangle^{\delta_0})}\right) \|v_{2^{-k}}\|^{\frac{\gamma}{2}}_{H^2(\mathbb{R}^2,\langle x\rangle^{-\delta'})} \\ &\leqslant e^{K_k} e^{(1+\|v_0\|^a_{H^2(\mathbb{R}^2,\langle x\rangle^{\delta_0})})(1+|\log\varepsilon|)^{2\sigma(1+\kappa)}} e^{\log(1+\|v\|_{H^\gamma(\mathbb{R}^2,\langle x\rangle^\delta)})^{2\sigma}},\end{aligned}$$

for some small $\delta' > 0$ and $\kappa > 0$. Note that for $s \in (0,1)$ and any $\kappa' > 0$ it holds $e^{\log(1+x)^s} \lesssim x^{\kappa'}$, so that we can close this estimate with an application of the Young product inequality (8.7) as

$$\|v_{2^{-k}}\|_{H^\gamma(\mathbb{R}^2,\langle x\rangle^\delta)} \leqslant e^{K_k} e^{(1+\|v_0\|^a_{H^2(\mathbb{R}^2,\langle x\rangle^{\delta_0})})(1+|\log\varepsilon|)^{2\sigma(1+\kappa)}}.$$

Reinserting this into the modified Brezis-Gallouet inequality, Lemma 8.5.1, yields

$$\|v_\varepsilon e^{-Y_\varepsilon}\|_{C([0,T];L^\infty(\mathbb{R}^2))} \lesssim K_\varepsilon + (1 + |\log\varepsilon|^{1+\kappa})(1 + \|v_0\|^a_{H^2(\mathbb{R}^2,\langle x\rangle^{\delta_0})}),$$

where we choose $\kappa > 0$ small enough such that $s := 2\sigma(1+\kappa) < 1$. Combining this estimate with Lemma 8.4.3 we end up with

$$\begin{aligned}\|v_{2^{-k}} - v_{2^{-k-1}}\|_{C([0,T];H^\gamma(\mathbb{R}^2,\langle x\rangle^\delta))} &\leqslant 2^{-k\kappa'} e^{K_k} e^{(1+\|v_0\|_{H^2(\mathbb{R}^2,\langle x\rangle^{\delta_0})})(1+|\log 2^{-k}|)^s} \\ &\lesssim_{v_0} e^{K_k} 2^{-k\kappa'/2}.\end{aligned}$$

for some $\kappa' > 0$ and where we used $s < 1$ in the second step. We therefore conclude that $v_{2^{-k}}$ is a Cauchy sequence whose limit $v \in C([0,T]; H^\gamma(\mathbb{R}^2, \langle x\rangle^\delta))$ solves (8.15). For the convergence of v_ε in probability to v and for uniqueness we proceed as in Theorem 8.4.4. □

Glossary

$\lesssim$	Means $\leqslant$ "up to a multiplicative, deterministic constant" 15
$\precsim$	Used for indices $i, j \in \mathbb{Z}$. Means $\leqslant$ "up to an additive, deterministic constant" 16
$\diamond$	Symbol used for Wick products or Skohorod integration 70
$\times$	Symbol used to connect products with factors in different lines 17
$\mathscr{A}$	Green's function of a Fourier multiplier $a(D)$ 131
$A_{\mathbb{N}^d}$	Shorthand for $A_{\mathbb{N}^d} = A + \lvert\mathbb{N}^d\rvert_{\mathfrak{s}}$, where A is the index set of a regularity structure $\mathscr{T} = (A, \mathcal{T}, G)$ 106
$\mathcal{B}^{\gamma}_{p,q,\mathfrak{s}}$	Besov space 28
$B_{\mathfrak{s}}(0,1)$	Scaled unit ball for the scaling vector $\mathfrak{s}$ 24
$\mathcal{C}^{\gamma}_{\mathfrak{s}}$	Hölder-Zygmund space 28
C^n_b	Functions with bounded derivatives up to order n 17
C^{∞}_{ω}	Ultra-differentiable functions 22
D	Symbol used for Fourier multipliers on $\mathbb{R}^d$ 130
$\mathcal{D}^{[\eta,\gamma]}$	Singular, modelled distributions 112
$\varDelta_j$	Littlewood-Paley block constructed from the dyadic partititon of unity φ_j 27

$\mathcal{D}^\gamma$	Space of modelled distributions 46
$\mathscr{E}^\varepsilon$	Extension from Bravais lattices $\mathcal{G}^\varepsilon$ to $\mathbb{R}^d$ 58
$\mathcal{E}_\Omega$	Whitney extension for modelled distributions 126
$\mathcal{F}_{\mathbb{R}^d}$	Fourier transform with convention: $\mathcal{F}_{\mathbb{R}^d} f(x) = \int \mathrm{d}\xi\, e^{2\pi\imath x\xi} f(\xi)$ 18
function like	Attribute for a sector of non-negative regularity 41
$\mathcal{G}$, $\mathcal{G}^\varepsilon$	Bravais lattices, $\mathcal{G}^\varepsilon = \varepsilon\cdot\mathcal{G}$ denotes the scaled lattice 50
$\widehat{\mathcal{G}}$	Fourier cell for a Bravais lattice $\mathcal{G}$ 50
$\Gamma^\alpha\tau$	Projection of $\Gamma\tau$ onto $\mathcal{T}_\alpha$ on some regularity structure $\mathscr{T} = (A, \mathcal{T}, G)$ 42
H^γ	Fractional Sobolev space 28
Isotropic	The scaling vector $\mathfrak{s}$ equals $(1, \ldots, 1)$ 24, 28
$L^p(\mathbb{R}^d, \rho)$	Weighted L^p space with the convention $\\|f\\|_{L^p(\mathbb{R}^d,\rho)} = \\|f \cdot \rho\\|_{L^p(\mathbb{R}^d)}$ 23
$\boldsymbol{\mu}(\omega)$	Set of jump measures for symmetric random walks 63
$\mathbb{N}$	Natural numbers including 0, $\mathbb{N} = \{0, 1, 2, \ldots\}$ 16
$\mathbb{N}^d_{<\gamma}$	Set of $k \in \mathbb{N}^d$ with $\|k\|_\mathfrak{s} < \gamma$ 29
$\mathbb{N}^d_{>\gamma}$	“Boundary” of $\mathbb{N}^d_{<\gamma}$ 29
$\boldsymbol{\omega}$	Set of functions ω^{pol}, $\omega^{\exp}_\sigma$ that classify weights 20
Ω^T	Denoting the set $(0, T) \times \mathbb{R}^{d-1}$ 111
Ω^T_t	Denoting the set $(t, T) \times \mathbb{R}^{d-1}$ for $t \in (0, T)$ 111
(Π, Γ)	Model on a regularity structure $\mathscr{T} = (A, \mathcal{T}, G)$ 43
$P(F, \Pi)$, $P(F, \Gamma^\alpha)$	Paraproducts on a regularity structure 103
φ_j	Dyadic partition unity 26

φ^λ	L^1 scaling of φ 40
Ψ^j	Fourier transform of of the dyadic partition function φ_j 26
$\Psi^{<j-1}$	Abbreviation for $\sum_{i<j}\Psi^i$ 26
$R^\gamma_{x;h}$	(Anisotropic) Taylor remainder, $R^\gamma_{x;h} = \mathrm{Id} - T^\gamma_{x;h}$ 29
$\boldsymbol{\rho}(\omega)$	The set of weights, whose growth/decay is controlled by $\omega \in \boldsymbol{\omega}$ 20
Structure condition	Condition on the components of a modelled distribution 104
$\mathfrak{s}$	Scaling vector 24
$\mathcal{S}_\omega$	Ultra-differentiable Schwartz functions 21
Spectral support	The support of the Fourier transform of an (ultra-)distribution 17
$\mathscr{T} = (A, \mathcal{T}, G)$	Regularity structure 41
$\overline{\mathscr{T}} = (\overline{A}, \overline{\mathcal{T}}, \overline{G})$	Polynomial regularity structure 45
τ^α, $\tau^{<\gamma}$	Projection of $\tau \in \mathcal{T}$ on subspaces for some regularity structure $\mathscr{T} = (A, \mathcal{T}, G)$ 42
$\tau^{\mathbf{1}}$	Coefficient of $\tau \in \mathcal{T}$ in front of $\mathbf{1}$ 42
$T^\gamma_{x;h}$	(Anisotropic) Taylor polynomial up to order γ 29
$\mathcal{V}\backslash\mathcal{W}$	Complement of a sector 41

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